

An equivariant index for proper actions I

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Abstract

Equivariant indices, taking values in group-theoretic objects, have previously been defined in cases where either the group acting or the orbit space of the action is compact. In this paper, we define an equivariant index without assuming the group or the orbit space to be compact. This allows us to generalise an index of deformed Dirac operators, defined for compact groups by Braverman. In parts II and III of this series, we explore some properties and applications of this index.

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1 Introduction

Background

Equivariant index theory has a long and successful history, with applications in various areas of geometry and representation theory. To set the stage, let G be a Lie group, acting properly on a manifold M . Let $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^- \rightarrow M$ be a G -equivariant, \mathbb{Z}_2 -graded, Hermitian vector bundle, and D an odd, self-adjoint, G -equivariant elliptic differential operator on \mathcal{E} . In the basic form of equivariant index theory, one assumes G and M to be compact. Then the kernel of D is finite-dimensional. Hence one can define the equivariant index of D as

$$\text{index}_G D := [\ker D^+] - [\ker D^-] \in R(G).$$

Here D^\pm is the restriction of D to sections of \mathcal{E}^\pm , and $R(G)$ is the representation ring of G , whose elements are formal differences of isomorphism classes of finite-dimensional representations.

Generalisations of equivariant index theory to noncompact manifolds or groups have been obtained in two distinct directions.

1. If M and G may be noncompact, but the orbit space M/G is compact (we then call the action *cocompact*), one can apply the *analytic assembly map* introduced by Kasparov [26] and used in the Baum–Connes conjecture [6]. This has been studied very intensively in the last few decades. Successes of this area of index theory include the description of the K-theory of group C^* -algebras as in the Baum–Connes and Connes–Kasparov conjectures, and applications to the Novikov

conjecture. Furthermore, Kasparov [27] generalised Atiyah's index of transversally elliptic operators to the cocompact case. Index formulas for other indices were proved in [32, 35]. On homogeneous spaces, important results were obtained in [4, 16].

2. If G is compact, then one can often define an equivariant index of a suitable deformation of D . For the *trivial* group, some, but by no means all, well-known results on index theory on noncompact manifolds include the ones in [2, 11, 12, 13, 14, 15, 17, 19, 28]. For nontrivial compact groups, a natural deformation of Dirac operators poses a technical challenge related to unboundedness of the anticommutator of the Dirac operator and the deformation term. This was solved by Braverman [9]. The resulting index, including other, equivalent definitions, was used with great success in geometric quantisation, see e.g. [25, 29, 31].

The techniques used in these two cases, where M/G or G is compact, are very different. If M/G is compact, then one can apply methods from K -theory and K -homology of C^* -algebras, while if G is compact, then suitable deformations, or assumptions on the behaviour of the operator towards infinity, lead to indices in the completed representation ring

$$\widehat{R}(G) = \text{Hom}_{\mathbb{Z}}(R(G), \mathbb{Z}),$$

which contains infinite direct sums of irreducible representations, with finite multiplicities. (Operator algebraic techniques are used in the treatments of Callias-type operators in [14, 28] for the trivial group, but those techniques do not apply to the operators we are interested in.) This difference in approaches probably is an important reason why so far, no equivariant index theory has been developed that applies in cases where both M/G and G may be noncompact. This would have the potential for applications in representation theory of noncompact Lie groups, via non-cocompact actions, for example on (co)tangent bundles to homogeneous spaces, or on coadjoint orbits of groups containing G .

We should point out that by an equivariant index, we mean an index taking values in an object defined purely in terms of G (such as $R(G)$ or $\widehat{R}(G)$ if G is compact). For example, the equivariant coarse index (see [20], among many references), has been shown to be relevant for many problems in the noncompact setting. But because it takes values in the K -theory group of the equivariant Roe algebra of M , it is not the kind of index we are looking for here.

	M/G compact, D transversally elliptic	M/G noncompact, D a deformed Dirac operator
G compact	Atiyah, 1974 [3]	Braverman, 2002 [9]
G noncompact	Kasparov, 2015 [27]	Theorem 3.12

Table 1: Special cases of the index

Furthermore, in cases where M/G and G are both noncompact, index theory has been developed in terms of G -invariant sections [10, 22]. This contains information about multiplicities of the trivial representation, but in a fundamental way, the techniques used cannot be used to treat nontrivial representations.

The main result

Our goal in this paper is to develop and apply equivariant index theory for proper actions by possibly noncompact groups, with possibly noncompact orbit spaces. Motivated by work by Kasparov [27] and Braverman [9], we define the notion of G -Fredholm operators. For such operators, we define an equivariant index that generalises an index of transversally elliptic operators defined by Kasparov in the cocompact case, and an index of deformed Dirac operators for actions by compact groups, developed by Braverman. See Table 1. The main result in this paper is that the index we introduce allows us to complete this table, by filling the bottom-right entry, see Theorem 3.12.

In the second part of this series [23], we study some properties and applications of the equivariant index of deformed Dirac operators. These include an induction property, relations with the analytic assembly map and an index used by Mathai and Zhang in [30], a notion of Dirac induction (as in the Connes–Kasparov conjecture) based on non-cocompact actions, and a *quantisation commutes with reduction* property.

In the third part [24], we consider Spin^c -Dirac operators. For semisimple Lie groups with discrete series representations, the equivariant index is then directly related to multiplicities of discrete series representations, in cases where the Riemannian metric has a certain product form. Furthermore, the invariant index studied in [10, 22] can be recovered from the equivariant index. This leads to quantisation commutes with reduction results for this invariant index and an index in terms of multiplicities of discrete series representations, and to Atiyah–Hirzebruch type vanishing results in the cocompact Spin case.

The G-index

We now give some more technical details of the definition of the index we use. Let $K < G$ be a maximal compact subgroup. Consider the crossed product C^* -algebra $C_0(G/K) \rtimes G$. If M/G is compact, and D is transversally elliptic, then Kasparov [27] showed that D defines a natural class in the K -homology group $KK(C_0(G/K) \rtimes G, \mathbb{C})$ of $C_0(G/K) \rtimes G$. The algebra $C_0(G/K) \rtimes G$ is Morita-equivalent to the group C^* -algebra C^*K , so that this index can be viewed as an element of

$$KK(C^*K, \mathbb{C}) \cong \widehat{R}(K).$$

Note that, even though the index can be identified with an element of $\widehat{R}(K)$, it depends on the action by the whole group G . (The identification $KK(C_0(G/K) \rtimes G, \mathbb{C}) \cong \widehat{R}(K)$ involves an induction procedure from K to G .)

On the other hand, suppose that D is a Dirac-type operator. Let $\psi: M \rightarrow \mathfrak{g}$ (with \mathfrak{g} the Lie algebra of G), be an equivariant map. It induces a vector field v^ψ , which at a point $m \in M$ takes the value

$$v_m^\psi := \left. \frac{d}{dt} \right|_{t=0} \exp(-t\psi(m)) \cdot m.$$

Then we have the *deformed Dirac operator*

$$D_\psi := D - \sqrt{-1}c(v^\psi).$$

Suppose that the set of zeroes of v^ψ is cocompact. If $G = K$ is compact, then Braverman [9] showed that such an operator has a well-defined equivariant index in $\widehat{R}(K)$, after rescaling the map ψ by a function with suitable growth behaviour. In this case, one has the direct equality $C_0(G/K) \rtimes G = C^*K$, and Braverman's index equals a natural class defined by D_ψ in $KK(C^*K, \mathbb{C})$.

Motivated by these two examples, we define an operator to be *G-Fredholm* if it defines a class in $KK(C_0(G/K) \rtimes G, \mathbb{C})$. This class is then its equivariant index, special cases of which were summarised in Table 1. In Theorem 3.12, we show that deformed Dirac operators are *G-Fredholm*, so that the bottom-right entry in the table can be filled. In Proposition 3.13, we show that the index of deformed Dirac operators is independent of choices made.

Overview

We start in Section 2, by reviewing some background material on K -homology and crossed product algebras. Then we define the index and state the main result in Section 3. This result is proved in Sections 4 and 5.

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2 Preliminaries

We start by reviewing some basic facts about K-homology, some KK-theory, and crossed product C^* -algebras, for the benefit of readers who are not familiar with these topics. Experts should feel free to skip this section, or have a brief look at our notation and conventions.

We mention an isomorphism in KK-theory defined by Morita equivalence. This can be used to identify the equivariant index defined in Section 3.1 with an object that does not involve K-homology and C^* -algebras. For details about K-homology, we refer to Higson and Roe's book [21]. For the more general KK-theory, see Chapter VIII of [7].

2.1 Analytic K-homology

Let A be a separable C^* -algebra. A *Kasparov* (A, \mathbb{C}) -cycle is a triple (\mathcal{H}, F, π) , where

- \mathcal{H} is a \mathbb{Z}_2 -graded, separable Hilbert space;
- $F \in \mathcal{B}(\mathcal{H})$ is odd with respect to the grading;
- $\pi : A \rightarrow \mathcal{B}(\mathcal{H})$ is a $*$ -homomorphism into the even operators,

such that for all $a \in A$, the operators

$$\pi(a)(F^2 - 1), \quad [F, \pi(a)] \quad \text{and} \quad \pi(a)(F^* - F) \quad (2.1)$$

on \mathcal{H} are compact.

Definition 2.1. A *unitary equivalence* between two Kasparov (A, \mathbb{C}) -cycles (\mathcal{H}, F, π) and (\mathcal{H}', F', π') is an even unitary isomorphism $\mathcal{H} \cong \mathcal{H}'$, which intertwines the representations π and π' of A and the operators F and F' .

Definition 2.2. Consider two Kasparov (A, \mathbb{C}) -cycles (\mathcal{H}, F_0, π) and (\mathcal{H}, F_1, π) . Let $[0, 1] \rightarrow \mathcal{B}(\mathcal{H})$, denoted by $t \mapsto F_t$, be a norm-continuous path of odd operators. Suppose that for all $t \in [0, 1]$, the triple (\mathcal{H}, F_t, π) is a Kasparov (A, \mathbb{C}) -cycle. Then (\mathcal{H}, F_0, π) and (\mathcal{H}, F_1, π) are *operator homotopic*.

Definition 2.3. The *K-homology* of A (in even degree) is the Abelian group $\mathrm{KK}(A, \mathbb{C})$ with one generator for every class of Kasparov (A, \mathbb{C}) -cycles with respect to the equivalence relation generated by unitary equivalence and operator homotopy, subject to the relation

$$[\mathcal{H}, F, \pi] + [\mathcal{H}', F', \pi'] = [\mathcal{H} \oplus \mathcal{H}', F \oplus F', \pi \oplus \pi'],$$

for all equivalence classes $[\mathcal{H}, F, \pi]$ and $[\mathcal{H}', F', \pi']$ of Kasparov (A, \mathbb{C}) -cycles (\mathcal{H}, F, π) and (\mathcal{H}', F', π') , respectively.

If the operators (2.1) are zero, then (\mathcal{H}, F, π) is called a *degenerate cycle*, and turns out to represent the zero element in $\mathrm{KK}(A, \mathbb{C})$.

More generally, if B is another C^* -algebra (assumed to be σ -unital to avoid technical difficulties), then one has the notion of a Kasparov (A, B) -cycle. These are defined as Kasparov (A, \mathbb{C}) -cycles, with the Hilbert space replaced by a right Hilbert B -module. Similarly to Definition 2.3, one obtains the Abelian group $\mathrm{KK}(A, B)$. If there is a group G acting on A and B in a suitable way, there is an equivariant version as well. We will denote G -equivariant KK -theory and K -homology by a superscript G .

There is also an odd version of KK -theory, where there is no \mathbb{Z}_2 -grading. We will write KK or KK_0 for even KK -theory and KK_1 for odd KK -theory, and KK_* for the direct sum of the two.

The KK -group $\mathrm{KK}_*(A, B)$ is covariantly functorial in the first entry, and contravariantly functorial in the second. If C is a third C^* -algebra, there is the Kasparov product

$$\mathrm{KK}_*(A, B) \times \mathrm{KK}_*(B, C) \xrightarrow{\otimes_B} \mathrm{KK}_*(A, C).$$

It is associative, and functorial in all natural senses.

Example 2.4. If $A = \mathbb{C}$, then

$$\mathrm{KK}_0(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z} \quad \text{and} \quad \mathrm{KK}_1(\mathbb{C}, \mathbb{C}) \cong 0.$$

Example 2.5. Let $A := C_0(M)$, for a smooth Riemannian manifold M . Let D be an elliptic, odd, self-adjoint, first order differential operator on a \mathbb{Z}_2 -graded, Hermitian vector bundle $\mathcal{E} \rightarrow M$. Let $\pi_M: C_0(M) \rightarrow \mathcal{B}(L^2(\mathcal{E}))$ be given by pointwise multiplication. Then the triple

$$\left(L^2(\mathcal{E}), \frac{D}{\sqrt{D^2 + 1}}, \pi_M \right) \tag{2.2}$$

is a Kasparov $(C_0(M), \mathbb{C})$ -cycle. If a group acts on M and \mathcal{E} , preserving all structure and the operator D , then this is an equivariant Kasparov cycle. Its class in the K -homology of $C_0(M)$ is denoted by $[D]$.

Lemma 2.6. *Let (\mathcal{H}, F, π) be a Kasparov (A, \mathbb{C}) -cycle. Suppose that there exists a self-adjoint, odd involution T on \mathcal{H} that commutes with the action π of A , and anticommutes with F . Then (\mathcal{H}, F, π) represent the zero element in $KK(A, \mathbb{C})$.*

Proof. The path (\mathcal{H}, F_t, π) with $F_t = \cos(2\pi t)F + \sin(2\pi t)T$, gives an operator homotopy from (\mathcal{H}, F, π) to a degenerate cycle. \square

2.2 Crossed product C^* -algebras

Let A be a C^* -algebra, and G a locally compact group. Fix a left Haar measure dg on G , and let $\delta_G: G \rightarrow \mathbb{R}_{\geq 0}$ be the modular function, i.e. $d(gg') = \delta_G(g')dg$ for all $g' \in G$. Suppose there is a homomorphism $G \rightarrow \text{Aut}(A)$, continuous with respect to pointwise convergence in norm. We will denote the image of an element $g \in G$ under this map by g . The crossed product $A \rtimes G$ is a completion of the $*$ -algebra $C_c(G, A)$, with the product and $*$ -operation

$$\begin{aligned} (\varphi\varphi')(g) &= \int_G \varphi(g')g'\varphi'(g'^{-1}g) dg'; \\ \varphi^*(g) &= \delta_G(g)^{-1}g\varphi(g^{-1})^*, \end{aligned}$$

for $\varphi, \varphi' \in C_c(G, A)$ and $g \in G$.

The norm in which the completion is taken is defined as follows. Consider a Hilbert space \mathcal{H} , a unitary representation $\pi_G: G \rightarrow U(\mathcal{H})$ and a $*$ -representation $\pi_A: A \rightarrow \mathcal{B}(\mathcal{H})$, such that for all $g \in G$ and $a \in A$,

$$\pi_G(g)\pi_A(a)\pi_G(g)^* = \pi_A(ga).$$

These define a $*$ -representation

$$\pi_{G,A}: C_c(G, A) \rightarrow \mathcal{B}(\mathcal{H}),$$

by

$$\pi_{G,A}(\varphi) = \int_G \pi_A(\varphi(g))\pi_G(g) dg, \quad (2.3)$$

for $\varphi \in C_c(G, A)$. For such φ , one has

$$\|\pi_{G,A}(\varphi)\|_{\mathcal{B}(\mathcal{H})} \leq \|\varphi\|_{L^1(G,A)}.$$

The norm on $A \rtimes G$ is given by

$$\|\varphi\|_{A \rtimes G} := \sup_{\mathcal{H}, \pi_G, \pi_A} \|\pi_{G,A}(\varphi)\|_{\mathcal{B}(\mathcal{H})},$$

where the supremum is taken over \mathcal{H} , π_G and π_A as above.

If B is another C^* -algebra with a continuous action by G as above, and if $\psi: A \rightarrow B$ is a G -equivariant $*$ -homomorphism, then we have the induced $*$ -homomorphism

$$\psi_G: A \rtimes G \rightarrow B \rtimes G,$$

given by

$$\psi_G(\varphi)(g) = \psi(\varphi(g)), \quad (2.4)$$

for $\varphi \in C_c(G, A)$ and $g \in G$, and extended continuously. In what follows, we will often work with the dense subalgebra $C_c(G, A)$ (or an even smaller dense subspace), rather than with the complete algebra $A \rtimes G$.

2.3 Group C^* -algebras

Group C^* -algebras are important special cases of crossed products.

Definition 2.7. Let $A = \mathbb{C}$, with the trivial action by G . Then $\mathbb{C} \rtimes G$ is the *maximal group C^* -algebra* C^*G of G . It equals the completion of the convolution algebra $C_c(G)$ with respect to the norm

$$\|\varphi\|_{\max} := \sup_{\mathcal{H}, \pi_G} \|\pi_G(\varphi)\|_{\mathcal{B}(\mathcal{H})}.$$

Here the supremum runs over all unitary representations π_G of G in Hilbert spaces \mathcal{H} . For such a representation π_G , we use the same notation for the $*$ -homomorphism $\pi_G: C_c(G) \rightarrow \mathcal{B}(\mathcal{H})$ given by

$$\pi_G(\varphi)v = \int_G \varphi(g)\pi_G(g)v \, dg, \quad (2.5)$$

for $\varphi \in C_c(G)$ and $v \in \mathcal{H}$.

The *reduced group C^* -algebra* C_r^*G is the closure

$$\overline{\lambda(C_c(G))} \subset \mathcal{B}(L^2(G)),$$

where λ denotes the left regular representation of G in $L^2(G)$; i.e. for all $\varphi \in C_c(G)$, the operator $\lambda(\varphi)$ is given by convolution by φ .

The K -homology group of the group C^* -algebra C^*K of a compact group K has a very explicit description. (For compact groups, the maximal and reduced C^* -algebras coincide.) Let V be any irreducible representation space of K . Let $\pi_K: C^*K \rightarrow \mathcal{B}(V)$ be given by continuous extension of (2.5) (for $G = K$). Consider the grading on V for which all of V is the even part. Then

the triple $(V, 0, \pi_K)$ is a Kasparov (C^*K, \mathbb{C}) -cycle. This procedure defines an isomorphism of Abelian groups

$$KK(C^*K, \mathbb{C}) \cong \widehat{R}(K). \quad (2.6)$$

Here

$$\widehat{R}(K) \cong \text{Hom}_{\mathbb{Z}}(R(K), \mathbb{Z})$$

is the completion of the character ring $R(K)$ obtained by allowing infinite linear combinations of irreducible representations, but with finite multiplicities.

The isomorphism (2.6) can be described explicitly for more general K-homology cycles.

Lemma 2.8. *Let \mathcal{H} be a \mathbb{Z}_2 -graded, separable Hilbert space, with a unitary representation of K . Let $F \in \mathcal{B}(\mathcal{H})$ be an odd, self-adjoint, K -equivariant operator, such that (\mathcal{H}, F, π_K) is a Kasparov (C^*K, \mathbb{C}) -cycle. Let F^{\pm} be the restrictions of F to the even and odd parts of \mathcal{H} . Then the representation spaces $\ker F^{\pm}$ of K define elements of $\widehat{R}(K)$, and under the isomorphism (2.6), we have*

$$[\mathcal{H}, F, \pi_K] = [\ker F^+] - [\ker F^-]. \quad (2.7)$$

Proof. Let $V \in \widehat{K}$, and consider the class $[V] \in KK(\mathbb{C}, C^*K)$. Then, since F is K -equivariant, Example 18.3.2(a) in [7] implies that

$$[V] \otimes_{C^*K} [\mathcal{H}, F, \pi_K] = [(V \otimes \mathcal{H})^K, 1_V \otimes F, 1] \in KK(\mathbb{C}, \mathbb{C}) = \mathbb{Z}, \quad (2.8)$$

where 1 denotes scalar multiplication by complex numbers. Hence the operator $(1_V \otimes F)^2 - 1$ is compact, so that $1_V \otimes F$ is Fredholm. So its kernel is finite-dimensional, and (2.8) equals

$$[\ker(1_V \otimes F), 0, 1] + [\ker(1_V \otimes F)^{\perp}, 1_V \otimes F, 1]. \quad (2.9)$$

Define the operator $\text{sgn}(F)$ by functional calculus. On $\ker(1_V \otimes F)^{\perp}$, the operator $1_V \otimes \text{sgn}(F)$ has the properties of the operator T in Lemma 2.6. (In particular, its square is the identity.) Hence the second term in (2.9) is zero. We conclude that

$$[V] \otimes_{C^*K} [\mathcal{H}, F, \pi_K] = [\ker(1_V \otimes F)^K, 0, 1] = [\ker F^+ : V] - [\ker F^- : V].$$

Therefore, the multiplicity of V in both sides of (2.7) is equal. \square

2.4 Morita equivalence

Let G be a locally compact group, with left Haar measure dg and modular function δ_G . Let $K \subset G$ be a closed subgroup, with Haar measure dk . We assume K is unimodular for simplicity; later K will always be compact.

The C^* -algebra $C_0(G/K)$ has a natural continuous action by G , given by

$$(g \cdot h)(g'K) = h(g^{-1}g'K),$$

for $g, g' \in G$ and $h \in C_0(G/K)$. The main examples of crossed products we will use are of the form $C_0(G/K) \rtimes G$. This C^* -algebra is Morita equivalent to the group C^* -algebra C^*K via a Hilbert C^*K -module defined as in Situation 10 in [33]. This is Green's imprimitivity theorem, see Proposition 3 on page 203 of [18].

The isomorphism

$$KK(C_0(G/K) \rtimes G, \mathbb{C}) \cong KK(C^*K, \mathbb{C}) \cong \hat{R}(K) \quad (2.10)$$

defined by Morita equivalence can be described very explicitly. Let $V \in \hat{K}$, and consider the representation

$$\pi_{C_0(G/K) \rtimes G}: C_0(G/K) \rtimes G \rightarrow \mathcal{B}((L^2(G) \otimes V)^K), \quad (2.11)$$

defined by

$$(\pi_{C_0(G/K) \rtimes G}(\varphi)\sigma)(g) = \int_G \varphi(g, g'K) \delta_G(g')^{1/2} \sigma(g'^{-1}g) dg',$$

for $\varphi \in C_c(G, C_0(G/K))$, $\sigma \in (L^2(G) \otimes V)^K$, and $g \in G$. (On pages 131/132 of [36], it is explained how different powers of the modular function δ_G can be used.)

Proposition 2.9. *For $V \in \hat{K}$, the triple*

$$((L^2(G) \otimes V)^K, 0, \pi_{C_0(G/K) \rtimes G})$$

is a Kasparov $(C_0(G/K) \rtimes G, \mathbb{C})$ -cycle. The map

$$\hat{R}(K) \rightarrow KK(C_0(G/K) \rtimes G, \mathbb{C})$$

given by $[V] \mapsto [(L^2(G) \otimes V)^K, 0, \pi_{C_0(G/K) \rtimes G}]$ is the isomorphism given by Morita equivalence.

This is a special case of Proposition 3.11 in [23]. This fact means that the isomorphism (2.10) is given by an induction procedure from K to G . It will not be used in the current paper, it is only included to make the isomorphism (2.10) more explicit. We therefore postpone its proof to [23].

3 The index and the main result

Let G be a Lie group, with finitely many connected components. Let $K < G$ be a maximal compact subgroup. We will define the notion of a G -Fredholm operator, for proper actions by G . Such an operator has an equivariant index in the K -homology group of the crossed product C^* -algebra $C_0(G/K) \rtimes G$, or, via the isomorphisms of Subsection 2.3 and 2.4, in $\widehat{R}(K)$.

One special case of this index is an index of transversally elliptic operators for cocompact actions studied by Kasparov [27]. This in turn generalises Atiyah's index of transversally elliptic operators [3] in the compact case. Another special case is Braverman's index of deformed Dirac operators [9], for compact groups. The main result in this paper is Theorem 3.12, which generalises Braverman's index to noncompact groups.

For the rest of this paper, we fix a proper, isometric action by G on a complete Riemannian manifold M . Where convenient, we will use the Riemannian metric to identify $T^*M \cong TM$. We denote the space of vector fields on M by $\mathcal{X}(M)$. Let $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^- \rightarrow M$ be a \mathbb{Z}_2 -graded, Hermitian vector bundle. Suppose the action by G lifts to \mathcal{E} , preserving the grading and the Hermitian metric.

3.1 The equivariant index

Since G acts properly on M , the differentiable version of Abels' theorem, on page 2 of [1], states that there is a smooth, equivariant map

$$p: M \rightarrow G/K.$$

This defines a $*$ -homomorphism

$$p^*: C_0(G/K) \rightarrow C_b(M),$$

which induces

$$p_G^*: C_0(G/K) \rtimes G \rightarrow C_b(M) \rtimes G,$$

as in (2.4). As in (2.3), the $*$ -representation of $C_b(M)$ on $L^2(\mathcal{E})$ by pointwise multiplication, and the unitary representation by G in $L^2(\mathcal{E})$, combine to a $*$ -representation

$$\pi_{G, C_b(M)}: C_b(M) \rtimes G \rightarrow \mathcal{B}(L^2(\mathcal{E})). \quad (3.1)$$

The representation

$$\pi_{G, G/K}^p := \pi_{G, C_b(M)} \circ p_G^*: C_0(G/K) \rtimes G \rightarrow \mathcal{B}(L^2(\mathcal{E}))$$

is given explicitly by

$$(\pi_{G,G/K}^p(\varphi)s)(m) = \int_G \varphi(g, p(m)) g \cdot (s(g^{-1}m)) \, dg,$$

for $\varphi \in C_c(G, C_0(G/K))$, $s \in L^2(\mathcal{E})$ and $m \in M$.

Let $F \in \mathcal{B}(L^2(\mathcal{E}))$ be an odd, self-adjoint, equivariant operator.

Definition 3.1. The operator F is *G-Fredholm* for p if the triple

$$(L^2(\mathcal{E}), F, \pi_{G,G/K}^p) \quad (3.2)$$

is a Kasparov $(C_0(G/K) \rtimes G, \mathbb{C})$ -cycle. Then the *equivariant index*, or *G-index*, for p of F is the class

$$\text{index}_G^p(F) \in KK(C_0(G/K) \rtimes G, \mathbb{C})$$

of the triple (3.2).

If F is G-Fredholm for all smooth, equivariant maps $p: M \rightarrow G/K$, then F is *G-Fredholm*.

The crossed product $C_0(G/K) \rtimes G$ contains the dense subspace $C_c^\infty(G) \otimes C_c^\infty(G/K)$. For $e \in C_c^\infty(G)$ and $h \in C_c^\infty(G/K)$, we have

$$\pi_{G,G/K}^p(e \otimes h) = p^* h \pi_G(e).$$

Here $p^* h \in C_b^\infty(M)$ is viewed as a pointwise multiplication operator, and $\pi_G(e)$ is defined by

$$\pi_G(e)s = \int_G e(g) g \cdot s \, dg, \quad (3.3)$$

for all $s \in L^2(\mathcal{E})$. So, as an equivalent definition, F is G-Fredholm for p if and only if for all $e \in C_c^\infty(G)$ and $h \in C_c^\infty(G/K)$, the operators $p^* h \pi_G(e)(F^2 - 1)$ and $[F, p^* h]$ on $L^2(\mathcal{E})$ are compact.

Lemma 3.2. *If F is G-Fredholm, then the class $\text{index}_G^p(F)$ is independent of p .*

Proof. For $j \in \{0, 1\}$, let $p_j: M \rightarrow G/K$ be smooth, equivariant maps. Since G/K is G-equivariantly contractible, there is a G-equivariant homotopy

$$(p_t: M \rightarrow G/K)_{t \in [0,1]}$$

connecting p_0 to p_1 . (The space G/K is a universal example of proper G-actions [6].) Set $\mathcal{H} := C([0, 1], L^2(\mathcal{E}))$, and define $\tilde{F} \in \mathcal{B}(\mathcal{H})$ by applying F after evaluating at a point in $[0, 1]$. Define

$$\tilde{\pi}_{G,G/K}: C_0(G/K) \rtimes G \rightarrow \mathcal{B}(\mathcal{H})$$

by

$$(\tilde{\pi}_{G,G/K}(\varphi)\tilde{s})(t) := (\pi_{G,G/K}^{p_t}(\varphi)\tilde{s})(t),$$

for $\varphi \in C_0(G/K) \rtimes G$, $\tilde{s} \in \tilde{\mathcal{H}}$ and $t \in [0, 1]$. Then, the triple $(\tilde{\mathcal{H}}, \tilde{F}, \tilde{\pi}_{G,G/K})$ defines a homotopy class (a ‘standard’ homotopy in the sense of Definition 17.2.2 in [7])

$$[\tilde{\mathcal{H}}, \tilde{F}, \tilde{\pi}_{G,G/K}] \in KK(C_0(G/K) \rtimes G, C([0, 1])),$$

so

$$[L^2(\mathcal{E}), F, \pi_{G,M}^{p_0}] = [L^2(\mathcal{E}), F, \pi_{G,M}^{p_1}] \in KK(C_0(G/K) \rtimes G, \mathbb{C}).$$

□

Because of this lemma, the following definition makes sense.

Definition 3.3. If $F \in \mathcal{B}(L^2(\mathcal{E}))$ is a G -Fredholm operator, then its G -index is the class

$$\text{index}_G(F) := \text{index}_G^p(F) \in KK(C_0(G/K) \rtimes G, \mathbb{C}),$$

for any smooth, equivariant map $p: M \rightarrow G/K$.

From now on, we will also write $\pi_{G,G/K} := \pi_{G,G/K}^p$ when a map p as above is given, and there is no danger of confusion.

Via the Morita equivalence isomorphism of Subsection 2.4, we can identify the G -index of a G -Fredholm operator F with an element of $\hat{R}(K)$. Furthermore, if G/K has an equivariant Spin structure, we can use the Dirac induction isomorphism

$$\text{D-Ind}_K^G: R(K) \xrightarrow{\cong} K_*(C_r^*G)$$

from the Connes–Kasparov conjecture (see (4.20) in [6]) and the universal coefficient theorem to identify

$$\hat{R}(K) = \text{Hom}_{\mathbb{Z}}(R(K), \mathbb{Z}) = \text{Hom}_{\mathbb{Z}}(K_*(C_r^*G), \mathbb{Z}) = KK(C_r^*G, \mathbb{C}).$$

In that way, the G -index takes values in the K -homology of C_r^*G . These identifications will be useful in some of the applications in [23]. Note that, while the G -index can be identified with an element of $\hat{R}(K)$, it depends on the action by the whole group G . This is apparent from the results in [23], where, for example, the G -index of certain operators is related to discrete series representations of semisimple groups that have such representations.

One special case of the G -index is an index of transversally elliptic operators for cocompact actions studied by Kasparov.

Theorem 3.4 (Kasparov). *Suppose M/G is compact, and let F be a properly supported, odd, self-adjoint, equivariant pseudo-differential operator on \mathcal{E} of order zero, which is transversally elliptic in the sense of Definition 6.1 in [27]. Then F is G -Fredholm.*

Proof. See Proposition 6.4 and Remark 8.19 in [27]. \square

In the setting of this result, the G -index of F is the index defined by Kasparov in Remark 8.19 in [27]. If M and G are compact, this reduces to Atiyah's index of transversally elliptic operators [3].

3.2 Differential operators

In the setting of Subsection 3.1, let D be an elliptic, self-adjoint, odd, equivariant, first order differential operator on \mathcal{E} . Let σ_D be its principal symbol. Set

$$F := \frac{D}{\sqrt{D^2 + 1}} \in \mathcal{B}(L^2(\mathcal{E})).$$

We will use the following criterion for F to be G -Fredholm in the proof of our main result, Theorem 3.12.

Lemma 3.5. *Suppose that for a smooth, equivariant map $p: M \rightarrow G/K$, and all $e \in C_c^\infty(G)$ and $h \in C_c^\infty(G/K)$, the operator*

$$(D^2 + 1)^{-1} \pi_G(e) p^* h \tag{3.4}$$

on $L^2(\mathcal{E})$ is compact. Then F is G -Fredholm for p .

Proof. This follows from Baaj and Julg's description of unbounded KK-theory in [5]. Indeed, the operator (3.4) is compact if and only if its adjoint

$$p^* h \pi_G(e^*) (D^2 + 1)^{-1} = p^* h \pi_G(e^*) (F^2 - 1)$$

is. Hence the second condition in Definition 2.1 in [5] holds. Furthermore, the commutator

$$[D, p^* h \pi_G(e)]$$

equals

$$\sigma_D(p^* dh) \pi_G(e).$$

This operator is bounded, so D satisfies all conditions in Definition 2.1 in [5]. The claim therefore follows from Proposition 2.2 in [5]. \square

The operator D is unbounded, so it is not the kind of operator to which Definition 3.1 applies. But we will say that D is G -Fredholm, or G -Fredholm for p , if the operator F has the respective property. Then the condition in Lemma 3.5 is sufficient for D to be G -Fredholm for p . If D is G -Fredholm for p , then we write

$$\text{index}_G^p(D) := \text{index}_G^p(F) \in \text{KK}(C_0(G/K) \rtimes G, \mathbb{C}).$$

In particular, in the setting of Lemma 3.5, we have the *spectral triple* $(C_c^\infty(G \times G/K), L^2(\mathcal{E}), D)$. If D is G -Fredholm (i.e. for all such maps p), we write

$$\text{index}_G(D) := \text{index}_G(F) \in \text{KK}(C_0(G/K) \rtimes G, \mathbb{C}).$$

If D is only essentially self-adjoint, then we define the G -Fredholm property for D , and its equivariant index, in terms of its self-adjoint closure.

Example 3.6. Suppose M/G is compact. Then the function p^*h is compactly supported for all $h \in C_c^\infty(G/K)$ and all smooth, equivariant maps $p: M \rightarrow G/K$. If D is elliptic, then the Rellich lemma therefore implies that for all $e \in C_c^\infty(G)$ and $h \in C_c^\infty(G/K)$, the operator

$$(D^2 + 1)^{-1} \pi_G(e) p^* h = \pi_G(e) (D^2 + 1)^{-1} p^* h$$

is compact. Hence D is G -Fredholm. This is a (trivial) special case of Proposition 6.4 and Remark 8.19 in [27], for transversally elliptic operators. In the elliptic case, this is analogous to the usual argument that the triple (2.2) is a Kasparov cycle.

This example shows that all challenges in investigating which elliptic differential operators are G -Fredholm come from cases where M/G is non-compact.

3.3 Deformed Dirac operators

Braverman [9] developed equivariant index theory of deformed Dirac operators for actions by compact groups on possibly noncompact manifolds. We will see that this theory fits into the framework of G -Fredholm operators, where it generalises to noncompact groups.

Let us define the deformed Dirac operators considered by Braverman. (They already played an important role on compact manifolds in [34].) Let M , G and \mathcal{E} be as in Subsection 3.1. From now on, we suppose there is a vector bundle homomorphism

$$c: TM \rightarrow \text{End}(\mathcal{E}),$$

whose image lies in the skew-adjoint, odd endomorphisms, such that for all $v \in TM$,

$$c(v)^2 = -\|v\|^2.$$

Then \mathcal{E} is called a *Clifford module* over M , and c is called the *Clifford action*.

A *Clifford connection* is a Hermitian connection $\nabla^\mathcal{E}$ on \mathcal{E} that preserves the grading on \mathcal{E} , such that for all vector fields $v, w \in \mathcal{X}(M)$,

$$[\nabla_v^\mathcal{E}, c(w)] = c(\nabla_v^{TM} w),$$

where ∇^{TM} is the Levi-Civita connection on TM . We will identify $TM \cong T^*M$ via the Riemannian metric. Then the Clifford action c defines a map

$$c: \Omega^1(M; \mathcal{E}) \rightarrow \Gamma^\infty(\mathcal{E}).$$

The *Dirac operator* D associated to a Clifford connection $\nabla^\mathcal{E}$ is defined as the composition

$$D: \Gamma^\infty(\mathcal{E}) \xrightarrow{\nabla^\mathcal{E}} \Omega^1(M; \mathcal{E}) \xrightarrow{c} \Gamma^\infty(\mathcal{E}). \quad (3.5)$$

In terms of a local orthonormal frame $\{e_1, \dots, e_{\dim M}\}$ of TM , one has

$$D = \sum_{j=1}^{\dim M} c(e_j) \nabla_{e_j}^\mathcal{E}. \quad (3.6)$$

This operator interchanges sections of \mathcal{E}^+ and \mathcal{E}^- . We will denote the restriction of D to $\Gamma^\infty(\mathcal{E}^\pm)$ by D^\pm .

Suppose that for all $g \in G$, $m \in M$, $v \in T_m M$ and $u \in \mathcal{E}_m$ we have¹

$$g \cdot c(v)u = c(g \cdot v)g \cdot u.$$

Then \mathcal{E} is called a *G-equivariant Clifford module* over M . In this case, the Dirac operator associated to a G -invariant Clifford connection is G -equivariant. We fix a G -invariant Clifford connection $\nabla^\mathcal{E}$ on \mathcal{E} for the rest of this paper, and consider the Dirac operator D associated to $\nabla^\mathcal{E}$.

If M and $G = K$ are *compact*, then the kernel of the Dirac operator D is finite-dimensional, and we have its equivariant index

$$\text{index}_K(D) = [\ker D^+] - [\ker D^-] \in R(K).$$

More generally, if M/G is compact, we can apply the analytic assembly map μ_M^G from [6] to the K -homology class $[D]$ as in Example 2.5, to obtain an index

$$\mu_M^G[D] \in KK(\mathbb{C}, C^*G).$$

¹In fact, this condition implies that the action by G preserves the Riemannian metric.

Our goal in this paper is to develop index theory for cases where both G and M/G are noncompact, however.

To define an index when M/G is noncompact, we consider a smooth, equivariant map

$$\psi: M \rightarrow \mathfrak{g}.$$

It induces a vector field $v^\psi \in \mathfrak{X}(M)$, defined by

$$v_m^\psi = \left. \frac{d}{dt} \right|_{t=0} \exp(-t\psi(m)) \cdot m,$$

for all $m \in M$. This vector field is G -invariant.

Definition 3.7. The *Dirac operator deformed by ψ* is the operator

$$D_\psi = D - \sqrt{-1}c(v^\psi)$$

on $\Gamma^\infty(\mathcal{E})$.

Let $\text{Zeroes}(v^\psi) \subset M$ be the set of zeroes of v^ψ .

Assumption 3.8. The set $\text{Zeroes}(v^\psi)$ is cocompact; i.e. $\text{Zeroes}(v^\psi)/G$ is compact.

If G is compact, Braverman defined an equivariant index of the Dirac operator deformed by $f\psi$, for a function f that is *admissible* in the following sense.

Definition 3.9. Let a real-valued function $\rho \in C^\infty(M)^G$ be given. A non-negative function $f \in C^\infty(M)^G$ is ρ -*admissible* if, outside a cocompact subset of M , we have

$$\frac{f^2}{\|df\| + f + 1} \geq \rho.$$

This property of a function f reflects that it grows fast enough compared to its derivative. (Braverman's notion of admissibility, as in Definition 2.6 in [9], is slightly different. The one we use is sufficient, however.) Admissible functions always exist.

Lemma 3.10. For any real-valued $\rho \in C^\infty(M)^G$, there exist ρ -admissible functions.

Proof. Without loss of generality, we may assume that $\rho \geq 1/4$. By Lemma C.3 in [22], there is a positive, G -invariant smooth function f such that

$$\begin{aligned} f^{-1} &\leq \rho^{-1}/4 \\ \|\mathbf{d}(f^{-1})\| &\leq \rho^{-1}/2. \end{aligned}$$

Then $f^{-1} \leq 1$, so $f^{-2} \leq f^{-1}$, and hence

$$\frac{f^2}{\|\mathbf{d}f\| + f + 1} = (\|\mathbf{d}(f^{-1})\| + f^{-1} + f^{-2})^{-1} \geq (\|\mathbf{d}(f^{-1})\| + 2f^{-1})^{-1} \geq \rho.$$

□

Theorem 3.11 (Braverman). *Suppose $G = K$ is compact. Then there is a real-valued function $\rho_{\text{Br}} \in C^\infty(M)^K$ such that for all ρ_{Br} -admissible functions $f \in C^\infty(M)^K$, and all irreducible representations V of K , the multiplicity m_V^\pm of V in*

$$\ker D_{f\psi} \cap L^2(\mathcal{E}^\pm)$$

is finite. The index

$$\text{index}_K^{\text{Br}}(D_{f\psi}) := \sum_{V \in \hat{K}} (m_V^+ - m_V^-) V \in \hat{R}(K)$$

is independent of the ρ_{Br} -admissible function f , the connection $\nabla^\mathcal{E}$ and the complete Riemannian metric on M .

This is Theorem 2.9 in [9]. Independence of the various choices follows from a general cobordism invariance result for this index, Theorem 3.7 in [9]. Note that if $G = K$, there is only one map $M \rightarrow G/K$.

Equivariant index theory of deformed Dirac operators is relevant for example to geometric quantisation. Already in the compact case, deformed Dirac operators were used by Tian and Zhang in [34] to obtain a localisation result for the index of a Dirac operator. For compact groups and noncompact manifolds, they were used in [25, 29, 31]. The indices used in [29, 31] are defined differently, but are equal to Braverman's. The fact that these three indices turn out to be equal is an indication that they are natural objects to study. Another reason why it is natural to consider deformed Dirac operators is that the deformation term arises from basic constructions in certain cases. For example, for Spin^c -Dirac operators, the deformation just amounts to a different choice of connection (see Remark 3.7 in [25]). More generally, it is interesting to investigate a class of equivariant elliptic operators for non-cocompact actions that have well-defined equivariant indices.

A complication for the deformed Dirac operators studied here is that the anticommutator $Dc(v^\psi) + c(v^\psi)D$ is not a vector bundle endomorphism of \mathcal{E} , but has a first order part (see Lemma 4.5). This is in contrast to Callias-type deformations of Dirac operators, see [2, 13, 14, 15, 28].

3.4 Noncompact groups; the main result

We now allow G to be noncompact. In [10, 22], one studies an index of deformed Dirac operators that only involves G -invariant sections of \mathcal{E} , i.e. the isotypical component of the trivial representation. It is an interesting and natural question if this can be extended to nontrivial representations. However, it is not clear a priori how to do this, or even where such an index should take values. For one thing, the unitary dual of a noncompact group is not discrete. For another, the nontrivial irreducible representations of a noncompact simple group are infinite-dimensional, which means Braverman's arguments in [9] do not apply to nontrivial representations.

The techniques used in [10, 22] fundamentally only apply to G -invariant sections, but the notion of G -Fredholm operators makes a completely different approach possible. This allows us to generalise Braverman's index to noncompact groups and nontrivial representations. This is the main result in this paper.

Theorem 3.12 (Deformed Dirac operators are G -Fredholm). *Let $p: M \rightarrow G/K$ be smooth and equivariant. There is a real-valued function $\rho \in C^\infty(M)^G$ (depending on the Riemannian metric on TM , the connection on \mathcal{E} used to define D , and the map p) such that the operator $D_{f\psi}$ is G -Fredholm for p , for all ρ -admissible functions f .*

In the setting of this result, we have the G -index

$$\text{index}_G^p(D_{f\psi}) \in KK(C_0(G/K) \rtimes G, \mathbb{C}),$$

for ρ -admissible functions f . This index is independent of the map p , the Riemannian metric on M , the function f , and the connection $\nabla^\mathcal{E}$. Because the function ρ depends on these data, we need to be somewhat careful in the precise formulation of this independence property.

Proposition 3.13. *For $j = 0, 1$, let a complete, G -invariant Riemannian metric B_j on TM be given, and a corresponding G -equivariant Clifford action c_j by TM on the vector bundle \mathcal{E} . Write \mathcal{E}_j for the vector bundle \mathcal{E} with this Clifford action. Fix a G -invariant Clifford connection $\nabla^{\mathcal{E}_j}$ on \mathcal{E}_j , and a smooth, equivariant map $p_j: M \rightarrow G/K$. Let $D^{\mathcal{E}_j}$ be the Dirac operator associated to the connection $\nabla^{\mathcal{E}_j}$.*

Let ρ_j be a function as in Theorem 3.12, for the data $(B_j, \mathcal{E}_j, \nabla^{\mathcal{E}_j}, p_j)$. Let f_j be a ρ_j -admissible function. Consider the deformed Dirac operator

$$D_{f_j\psi}^{\mathcal{E}_j} = D^{\mathcal{E}_j} - \sqrt{-1}f_j c_j(v^\psi).$$

Then

$$\text{index}_G^{p_0}(D_{f_0\psi}^{\mathcal{E}_0}) = \text{index}_G^{p_1}(D_{f_1\psi}^{\mathcal{E}_1}) \in \text{KK}(C_0(G/K) \rtimes G, \mathbb{C}).$$

We write

$$\text{index}_G(\mathcal{E}, \psi) := \text{index}_G^p(D_{f\psi}),$$

for any complete, G -invariant Riemannian metric on TM , G -invariant Clifford connection, smooth equivariant map $p: M \rightarrow G/K$ and ρ -admissible function f . Because of Lemma 2.8, this index reduces to Braverman's index if G is compact.

As far as the authors know, there is currently no other version of equivariant index theory for noncompact groups and orbit spaces. Here we interpret an equivariant index as taking values in an object defined purely in terms of the group acting, such as $\text{KK}(C_0(G/K) \rtimes G, \mathbb{C})$. In the induction result in [23], we give an explicit description of the image of the G -index of a deformed Dirac operator in $\widehat{R}(K)$. In [23], we also give some relations between the G -index and existing indices in cases where M/G is compact, and some properties and applications in the general case.

3.5 Idea of the proof

We prove Theorem 3.12 and Proposition 3.13 in Sections 4 and 5. Here we describe the idea of the proof of Theorem 3.12.

First of all, it is important that the operator $D_{f\psi}$ is essentially self-adjoint. This follows from Proposition 10.2.11 in [21], which we restate here.

Proposition 3.14. *If A is a symmetric, first order differential operator on $\mathcal{E} \rightarrow M$, with principal symbol σ_A , and if its propagation speed*

$$\sup\{\|\sigma_A(v)\|; v \in TM \text{ and } \|v\| = 1\}$$

is finite, then A is essentially self-adjoint.

To apply Lemma 3.5, we choose an open cover $\{U_j\}_{j=0}^\infty$ of M by G -invariant, relatively cocompact open sets U_j . We choose these sets so that for all j , the boundary ∂U_j is a smooth submanifold of M , and has a neighbourhood in

U_j diffeomorphic to $\partial U_j \times [0, 1[$. Because $\text{Zeroes}(v^\psi)/G$ is compact, we can choose this cover so that $\|v^\psi\|$ has a positive lower bound on U_j for all $j \geq 1$. Then in particular, $\text{Zeroes}(v^\psi) \subset U_0$. In addition, we choose this cover so that every point in M is contained in only finitely many of the sets U_j .

Let $\{\chi_j\}_{j=0}^\infty$ be a sequence of G -invariant functions such that $\text{supp}(\chi_j) \subset U_j$ for all j , and $\{\chi_j^2\}_{j=0}^\infty$ is a partition of unity. Let $p: M \rightarrow G/K$ be a smooth, equivariant map. Then for all $f \in C^\infty(M)^G$, and all $e \in C_c^\infty(G)$ and $h \in C_c^\infty(G/K)$, we have

$$(D_{f\psi}^2 + 1)^{-1} \pi_G(e) p^* h = \sum_{j=0}^{\infty} (D_{f\psi}^2 + 1)^{-1} \pi_G(e) p^* h \chi_j^2. \quad (3.7)$$

Because the function $p^* h \chi_j^2$ has compact support for all j , the Rellich lemma implies that all terms in the sum on the right hand side are compact operators. Therefore, the operator $D_{f\psi}$ is G -Fredholm if f is ρ -admissible, for ρ as in the following proposition.

Proposition 3.15. *There is a real-valued function $\rho \in C^\infty(M)^G$, such that for all $e \in C_c^\infty(G)$ and all $h \in C_c^\infty(G/K)$, there is a constant $B_{e,h}$, such that for all ρ -admissible functions f , and all $j \geq 1$,*

$$\|(D_{f\psi}^2 + 1)^{-1} \pi_G(e) p^* h \chi_j^2\| \leq 2^{-j} B_{e,h}. \quad (3.8)$$

If f is ρ -admissible for such a function ρ , then the sum (3.7) of compact operators converges in the operator norm, to a compact operator. Hence $D_{f\psi}$ is G -Fredholm, by Lemma 3.5.

The idea behind the proof of Proposition 3.15 is to show that we have

$$D_{f\psi}^2 \pi_G(e) p^* h \chi_j^2 = (D^2 + f^2 \|v^\psi\|^2) \pi_G(e) p^* h \chi_j^2 + A_j$$

where A_j is a bounded operator. For $j \geq 1$, the term $f^2 \|v^\psi\|^2$ is large on U_j if f is large there, so that the norm on the left hand side of (3.8) is small in an appropriate sense. Making this idea precise turns out to require a more elaborate argument than the authors had expected initially.

It is important that the function ρ does not depend on e and h in Proposition 3.15. For this reason, we will need to carefully distinguish between constants depending on e and h , and constants depending on other data, in the estimates in Sections 4 and 5.

4 Decomposing Dirac operators

This section contains some preparatory material for the proof of Proposition 3.15 in Section 5. We will decompose the square of a deformed Dirac operator, and use the fact that some terms in this decomposition only differentiate in orbit directions. These terms are *G-differential operators* in the sense of Subsection 4.1, which means we can apply a general estimate for such operators. We will also use embeddings of open subsets of M into complete manifolds, as discussed in Subsection 4.4. This will make certain locally defined operators invertible.

Fix a smooth, equivariant map $p: M \rightarrow G/K$. It is automatically a submersion, so that

$$N := p^{-1}(eK) \subset M$$

is a smooth, K -invariant submanifold. Furthermore, the map $G \times N \rightarrow M$, mapping an element $(g, n) \in G \times N$ to gn , descends to a G -equivariant diffeomorphism

$$G \times_K N \xrightarrow{\cong} M. \quad (4.1)$$

We also fix an equivariant map $\psi: M \rightarrow \mathfrak{g}$ for which $\text{Zeroes}(v^\psi)/G$ is compact, and a real-valued function $f \in C^\infty(M)^G$. Let $D_{f\psi}$ be the deformed Dirac operator as in Definition 3.7.

4.1 G-differential operators

Before analysing deformed Dirac operators, we obtain an estimate for operators whose highest-order parts only differentiate in orbit directions. In this subsection and the next, $\mathcal{E} \rightarrow M$ is a G -equivariant, Hermitian vector bundle as before, but we will not use the Clifford action for now.

For an element $X \in \mathfrak{g}$, we denote the induced vector field on M by X^M . Our sign convention is that for all $m \in M$,

$$X_m^M = \left. \frac{d}{dt} \right|_{t=0} \exp(-tX) \cdot m.$$

The Lie derivative of sections of \mathcal{E} with respect to X is denoted by \mathcal{L}_X . Let $\tau_M: T^*M \rightarrow M$ be the cotangent bundle projection.

Let $T_G^*M \subset T^*M$ be the subset of elements that annihilate tangent vectors to orbits:

$$T_G^*M = \{ \xi \in T^*M; \langle \xi, X_{\tau_M(\xi)}^M \rangle = 0 \text{ for all } X \in \mathfrak{g} \}.$$

If G is compact, a differential operator is transversally elliptic if its principal symbol is invertible outside a compact subset of T_G^*M . In a sense, G -differential operators have the opposite property.

Definition 4.1. A differential operator A on $\Gamma^\infty(\mathcal{E})$, with principal symbol σ_A , is a G -differential operator if σ_A is zero on T_G^*M .

It follows immediately that, if G is compact, then for any transversally elliptic operator B and any G -differential operator A , the operator $A + B$ has the same symbol class in K -theory as B :

$$[\sigma_{A+B}] = [\sigma_B] \in K^0(T_G^*M).$$

A G -differential operator of order at least 2 can have lower-order terms that do not just differentiate in orbit directions. We will only consider first order G -differential operators, however, and these can be described entirely in terms of differentiation along orbits. Fix a basis $\{X_1, \dots, X_{\dim G}\}$ of \mathfrak{g} .

Lemma 4.2. A first order differential operator on \mathcal{E} is a G -differential operator if and only if there are vector bundle endomorphisms a_j and b of \mathcal{E} such that

$$A = \sum_{j=1}^{\dim G} a_j \mathcal{L}_{X_j} + b. \quad (4.2)$$

Proof. Let A be a first order differential operator on \mathcal{E} . If A is of the form (4.2), then for all $m \in M$ and $\xi \in (T_G^*M)_m$,

$$\sigma_A(\xi) = \sum_{j=1}^{\dim G} (a_j)_m \langle \xi, (X_j^M)_m \rangle = 0.$$

Conversely, suppose A is a G -differential operator. For all j , let $(X_j^M)^* \in \Omega^1(M)$ be dual to X_j^M with respect to the Riemannian metric. Define the differential operator

$$\tilde{A} := \sum_{j=1}^{\dim G} \sigma_A((X_j^M)^*) \mathcal{L}_{X_j}$$

on \mathcal{E} . Then for all $m \in M$ and $\xi \in T^*M$,

$$\sigma_{\tilde{A}}(\xi) = \sigma_A((X_j^M)^*)_m \langle \xi, (X_j^M)_m \rangle = \sigma_A(\xi).$$

Hence $b := A - \tilde{A}$ is a vector bundle endomorphism. \square

4.2 An estimate for G-differential operators

One ingredient of the proof of Proposition 3.15 is an estimate for G-equivariant, first order G-differential operators. In the proof of this estimate, we will use certain compact subsets of G.

Lemma 4.3. *For all $e \in C_c^\infty(G)$ and $h \in C_c^\infty(G/K)$, there is a compact subset $S_{e,h} \subset G$, independent of the map p , such that for all $\tilde{e} \in C_c^\infty(G)$ with support inside $\text{supp } e$, and all $s \in \Gamma(\mathcal{E})$, $g \in G \setminus S_{e,h}$ and $n \in N$, we have*

$$(\pi_G(\tilde{e})p^*h s)(gn) = 0.$$

Proof. Let $e \in C_c^\infty(G)$ and $h \in C_c^\infty(G/K)$ be given. Let $q: G \rightarrow G/K$ be the quotient map. Set

$$S_{e,h} := \{g \in G; \text{supp}(e) \cap g(q^{-1}(\text{supp}(h)))^{-1} \neq \emptyset\}.$$

Then, if $\tilde{e} \in C_c^\infty(G)$ is supported in $\text{supp } e$, we have for all $s \in \Gamma(\mathcal{E})$, $g \in G$ and $n \in N$,

$$(\pi_G(\tilde{e})p^*h s)(gn) = \int_G \tilde{e}(g')h(g'^{-1}gK)g'(s(g'^{-1}gn)) dg'.$$

If $g \notin S_{e,h}$, then $\tilde{e}(g')h(g'^{-1}gK) = 0$ for all $g' \in G$. □

For all $X \in \mathfrak{g}$, let L_X be the operator on $C^\infty(G)$ defined by the infinitesimal left regular representation. Then for all $e \in C_c^\infty(G)$,

$$\mathcal{L}_X \circ \pi_G(e) = \pi_G(L_X(e)). \quad (4.3)$$

This will be used to prove the estimate for G-differential operators we need.

Proposition 4.4. *For any $e \in C_c^\infty(G)$ and any $h \in C_c^\infty(G/K)$, there is a constant $B_{e,h} > 0$, such that for any G-equivariant, first order G-differential operator A on \mathcal{E} , with a_j and b as in Lemma 4.2 bounded on N , there is a constant $C_{A,p} > 0$, independent of e and h , such that the operator*

$$A\pi_G(e)p^*h$$

on $L^2(\mathcal{E})$ is bounded, with norm at most $B_{e,h}C_{A,p}$.

Proof. Let $e \in C_c^\infty(G)$ and $h \in C_c^\infty(G/K)$ be given, and let $S_{e,h} \subset G$ be as in Lemma 4.3. For $j, k = 1, \dots, \dim G$, let $\text{Ad}_j^k \in C^\infty(G)$ be the functions such that for all j , and all $g \in G$,

$$\text{Ad}(g)X_j = \sum_{k=1}^{\dim G} \text{Ad}_j^k(g)X_k.$$

Set

$$\|\text{Ad}\|_{S_{e,h}} := \max_{g \in S_{e,h}} \max_{j,k=1,\dots,\dim G} |\text{Ad}_j^k(g)|,$$

and

$$B_{e,h} := \max \left\{ \|e\|_{L^1(G)} \|h\|_\infty, \dim G \cdot \|\text{Ad}\|_{S_{e,h}} \cdot \sum_{j=1}^{\dim G} \|L_{X_j}(e)\|_{L^1(G)} \|h\|_\infty \right\}.$$

Let A be a G -equivariant G -differential operator. Write

$$A = \sum_{j=1}^{\dim G} a_j \mathcal{L}_{X_j} + b$$

as in Lemma 4.2. By assumption, the pointwise norms of the vector bundle endomorphisms a_j and b are bounded on N . Set

$$C_{a,p} := \sum_{j=1}^{\dim G} \sup_{n \in N} \|(a_j)_n\|;$$

$$C_{b,p} := \sup_{n \in N} \|b_n\|;$$

$$C_{A,p} := C_{a,p} + C_{b,p}.$$

The operators a_j are not G -equivariant in general, even though the whole operator A is. Let $s \in \Gamma_c^\infty(\mathcal{E})$, $g \in G$ and $n \in N$ be given. Then equivariance of A implies that

$$\begin{aligned} (g^{-1}As)(n) &= (Ag^{-1}s)(n) \\ &= \sum_{j=1}^{\dim G} (a_j)_n g^{-1}(\mathcal{L}_{\text{Ad}(g)X_j} s(gn)) + b_n g^{-1}(s(gn)). \end{aligned}$$

So

$$(As)(gn) = \sum_{j=1}^{\dim G} (g(a_j)_n g^{-1})(\mathcal{L}_{\text{Ad}(g)X_j} s(gn)) + (gb_n g^{-1})s(gn).$$

Replacing s by $\pi_G(e)p^*hs$, we obtain the pointwise estimate

$$\begin{aligned} \|(A\pi_G(e)p^*hs)(gn)\| &\leq \sum_{j=1}^{\dim G} \|(\alpha_j)_n\| \|\mathcal{L}_{\text{Ad}(g)X_j} \pi_G(e)p^*hs(gn)\| + \|b_n\| \|\pi_G(e)p^*hs(gn)\| \\ &\leq C_{a,p} \sum_{j=1}^{\dim G} \|\mathcal{L}_{\text{Ad}(g)X_j} \pi_G(e)p^*hs(gn)\| + C_{b,p} \|e\|_{L^1(G)} \|h\|_\infty \|s(gn)\|. \end{aligned}$$

For every j , we have

$$\begin{aligned} \|\mathcal{L}_{\text{Ad}(g)X_j} \pi_G(e)p^*hs(gn)\| &\leq \sum_{k=1}^{\dim G} |\text{Ad}_j^k(g)| \|\mathcal{L}_{X_k} \pi_G(e)p^*hs(gn)\| \\ &\leq \sum_{k=1}^{\dim G} |\text{Ad}_j^k(g)| \|\pi_G(L_{X_k}(e))p^*hs(gn)\|, \end{aligned}$$

where we used (4.3).

The pointwise estimates obtained so far imply that

$$\begin{aligned} &\|A\pi_G(e)p^*hs\|_{L^2(\mathcal{E})} \\ &\leq C_{a,p} \dim(G) \|\text{Ad}\|_{S_{e,h}} \sum_{k=1}^{\dim G} \|\pi_G(L_{X_k}(e))p^*hs\|_{L^2(\mathcal{E})} + C_{b,p} \|e\|_{L^1(G)} \|h\|_\infty \|s\|_{L^2(\mathcal{E})} \\ &\leq B_{e,h} C_{A,p} \|s\|_{L^2(\mathcal{E})}. \end{aligned}$$

□

4.3 The square of a deformed Dirac operator

The first two steps in the proof of Proposition 3.15 are a decomposition of the square of the deformed Dirac operator $D_{f\psi}$, and a decomposition of the undeformed Dirac operator D . In both of these decompositions, G-differential operators appear. In Section 5, we will apply Proposition 4.4 to those operators.

Lemma 4.5. *For any local orthonormal frame $\{e_1, \dots, e_{\dim M}\}$ of TM , we have, locally,*

$$D_{f\psi}^2 = D^2 + f^2 \|v^\psi\|^2 - \sqrt{-1} \sum_{j=1}^{\dim M} c(e_j) c(\nabla_{e_j}^{TM} f v^\psi) + 2\sqrt{-1} f \nabla_{v^\psi}^\mathcal{E}.$$

Proof. We have

$$D_{f\psi}^2 = D^2 + f^2 \|\psi\|^2 - \sqrt{-1}(Dc(f\psi) + c(f\psi)D),$$

and, locally,

$$Dc(f\psi) + c(f\psi)D = \sum_{j=1}^{\dim M} c(e_j)c(\nabla_{e_j}^{\text{TM}} f\psi) - 2f\nabla_{\psi}^{\varepsilon}.$$

□

It will be important that in this expression for $D_{f\psi}^2$, the only first order term, $2f\nabla_{\psi}^{\varepsilon}$, is a G -differential operator.

Next, recall that we have

$$M \cong G \times_K N$$

as in (4.1). We have a G -equivariant isomorphism of vector bundles

$$TM \cong p^*T(G/K) \oplus G \times_K TN. \quad (4.4)$$

This decomposition of TM yields two projections

$$\begin{aligned} p_{G/K}: TM &\rightarrow p^*T(G/K); \\ p_N: TM &\rightarrow G \times_K TN. \end{aligned} \quad (4.5)$$

Identifying $T^*M \cong TM$ via the Riemannian metric as before, we obtain two partial Dirac operators

$$\begin{aligned} D_{G/K}: \Gamma^\infty(\mathcal{E}) &\xrightarrow{\nabla^\varepsilon} \Gamma^\infty(TM \otimes \mathcal{E}) \xrightarrow{p_{G/K} \otimes 1_\mathcal{E}} \Gamma^\infty(p^*T(G/K) \otimes \mathcal{E}) \xrightarrow{c} \Gamma^\infty(\mathcal{E}); \\ D_N: \Gamma^\infty(\mathcal{E}) &\xrightarrow{\nabla^\varepsilon} \Gamma^\infty(TM \otimes \mathcal{E}) \xrightarrow{p_N \otimes 1_\mathcal{E}} \Gamma^\infty(G \times_K TN \otimes \mathcal{E}) \xrightarrow{c} \Gamma^\infty(\mathcal{E}). \end{aligned}$$

Since $p_{G/K} + p_N$ is the identity map on TM , we have

$$D = D_{G/K} + D_N. \quad (4.6)$$

This decomposition will be useful to us, because $D_{G/K}$ is a G -differential operator, while D_N commutes with p^*h for all $h \in C_c^\infty(G/K)$.

Combining Lemma 4.5 and the decomposition (4.6) of D , we obtain an equality that we will use in our estimates. To state this equality, let $D_{G/K}^*$ and D_N^* be the formal adjoints of $D_{G/K}$ and D_N , respectively, with respect

the the L^2 -inner product. (Note that these operators are not symmetric in general.) Consider the following operators on $\Gamma^\infty(\mathcal{E})$:

$$\begin{aligned}
A_1 &:= -D_{f\psi} D_{G/K}^*; \\
A_2 &:= -D_{G/K} D_N^*; \\
A_3 &:= -\sqrt{-1} f c(v^\psi) D_{G/K}^*; \\
A_4 &:= \sqrt{-1} \sum_{j=1}^{\dim M} c(e_j) c(\nabla_{e_j}^{TM} f v^\psi); \\
A_5 &:= -2\sqrt{-1} f \nabla_{v^\psi}^\mathcal{E},
\end{aligned} \tag{4.7}$$

and

$$\Delta := A_1 + A_2 + A_3 + A_4 + A_5. \tag{4.8}$$

Lemma 4.6. *We have*

$$(D_{f\psi}^2 + 1)^{-1} - (D_N D_N^* + f^2 \|v^\psi\|^2 + 1)^{-1} = (D_{f\psi}^2 + 1)^{-1} \Delta (D_N D_N^* + f^2 \|v^\psi\|^2 + 1)^{-1}.$$

Proof. For any two invertible elements a and b of a ring, we have

$$a^{-1} - b^{-1} = a^{-1}(b - a)b^{-1}.$$

Hence the claim follows from Lemma 4.5 and (4.6). \square

4.4 Embeddings into complete manifolds

Let $U \subset M$ be a relatively cocompact, G -invariant open subset, on which $\|v^\psi\|$ has a positive lower bound. (We will apply what follows to the sets U_j in Subsection 3.5, for $j \geq 1$.) We would like to compare the restriction of the operator $(D_{f\psi}^2 + 1)^{-1}$ to such sets U to operators defined entirely in terms of data on U . But since $(D_{f\psi}^2 + 1)^{-1}$ is not a local operator, it does not restrict to open sets. Similarly, operators defined only on U may not be invertible, if U is not complete. For that reason, we embed U into a complete manifold, in the way we will describe now.

Suppose that ∂U is smooth, and that a K -invariant neighbourhood W_N of $\partial U \cap N$ in $U \cap N$ is K -equivariantly isometric to

$$(\partial U \cap N) \times]-\delta, 0[, \tag{4.9}$$

for a $\delta > 0$. Then

$$W := G \cdot W_N \cong G \times_K W_N$$

is a G -invariant neighbourhood of ∂U in U , and there is a G -equivariant isometry

$$W \cong \partial U \times]-\delta, 0[.$$

By glueing the “cylinder” $\partial U \times]-\delta, \infty[$ to U via this identification, we obtain a manifold V . The product of the restricted Riemannian metric from TM to $T(\partial U)$ and the Euclidean metric on $T(]-\delta, \infty[)$ defines a G -invariant Riemannian metric on $T(\partial U \times]-\delta, \infty[)$. This extends to TV , and makes V complete.

Define the K -invariant submanifold $N_V \subset V$ by attaching $(\partial U \cap N) \times]-\delta, \infty[$ to $N \cap U$ by identifying W_N with (4.9). Then

$$V \cong G \times_K N_V.$$

Let the G -equivariant smooth map $p_V: V \rightarrow G/K$ be given by $p_V(gn) = gK$, for $g \in G$ and $n \in N_V$. Then $N_V = p_V^{-1}(eK)$.

The vector bundle $\mathcal{E}|_U \rightarrow U$, the Clifford action by TU on it, and the Dirac operator $D|_U$ all extend to V . See e.g. Section 25 of [8]. We denote the extended vector bundle and Dirac operator by \mathcal{E}_V and D_V , respectively.

We will extend the map $\psi|_U: U \rightarrow \mathfrak{g}$ to a map $\psi_V: V \rightarrow \mathfrak{g}$, in such a way that $\|\psi_V\|$ has a positive lower bound on V . If we only needed a continuous extension, we could use the map $\tilde{\psi}_V: V \rightarrow \mathfrak{g}$, given by

$$\tilde{\psi}_V(y) = \begin{cases} \psi(y) & \text{if } y \in U; \\ \psi(x) & \text{if } y = (x, t) \in \partial U \times [0, \infty[. \end{cases}$$

The induced vector field ψ_V is continuous, and its norm has the same lower bound on V as ψ has on U . To obtain a smooth version, we use the following fact. Fix any K -invariant norm $\|\cdot\|$ on \mathfrak{g} .

Lemma 4.7. *There is an $\varepsilon > 0$, such that for all G -equivariant, continuous maps $\psi': \bar{U} \rightarrow \mathfrak{g}$ such that $\|\psi'(n) - \psi(n)\| \leq \varepsilon$ for all $n \in \bar{U} \cap N$, the norm of the vector field ψ' on \bar{U} has a positive lower bound.*

Proof. Let $\psi': \bar{U} \rightarrow \mathfrak{g}$ be a G -equivariant, continuous map. Since the norm $\|\psi'\|$ is G -invariant, it has a positive lower bound on \bar{U} precisely if it has one on $\bar{U} \cap N$. And since $\bar{U} \cap N$ is compact, $\|\psi'\|$ has a positive lower bound on this set precisely if it does not vanish there.

The set

$$\{(n, X) \in (\bar{U} \cap N) \times \mathfrak{g}; X \neq \psi_n\}$$

is open in $(\bar{U} \cap N) \times \mathfrak{g}$. Since $(n, \psi(n))$ is in this set for all $n \in \bar{U} \cap N$, and $\bar{U} \cap N$ is compact, there is an $\varepsilon > 0$ such that for all $n \in \bar{U} \cap N$ and $X \in \mathfrak{g}$ for

which $\|X - \psi(n)\| \leq \varepsilon$, we have $X_n^M \neq 0$. Hence if $\|\psi' - \psi\| \leq \varepsilon$ on $\bar{U} \cap N$, the claim follows. \square

Fix $\varepsilon > 0$ as in Lemma 4.7. Let $W'_N \subset W_N$ be a K -invariant neighbourhood of $\partial U \cap N$ such that for all $n = (x, t) \in W'_N \subset (\partial U \cap N) \times]-\delta, 0[$,

$$\|\psi(n) - \psi(x)\| \leq \varepsilon.$$

Let $\chi \in C^\infty(N_V)^K$ be a function with values in $[0, 1]$, such that

$$\begin{aligned} \chi &\equiv 1 \text{ on } (U \cap N) \setminus W'_N; \\ \chi &\equiv 0 \text{ on } \partial U \times]0, \infty[. \end{aligned}$$

Define $\psi_V: V \rightarrow \mathfrak{g}$ by

$$\psi_V(n) = \begin{cases} \psi(n) & \text{if } n \in (U \cap N) \setminus W'_N; \\ \chi(n)\psi(n) + (1 - \chi(n))\psi(x) & \text{if } n = (x, t) \in (\partial U \cap N) \times]-\delta, \infty[, \end{cases}$$

and extended G -equivariantly to V . Then ψ_V is smooth and G -equivariant, and $\|\psi'(n) - \psi(n)\| \leq \varepsilon$ for all $n \in \bar{U} \cap N$. Hence $\|v^{\psi_V}\|$ has a positive lower bound on U by Lemma 4.7. And if $(x, t) \in \partial U \times [0, \infty[$, then

$$v_{(x,t)}^{\psi_V} = (v_x^\psi, 0) \in T_x(\partial U) \times T_t(]-\delta, \infty[),$$

and this also has a positive lower bound as (x, t) ranges over $\partial U \times [0, \infty[$.

Let $f_V \in C^\infty(V)^G$ be any real-valued function such that $f_V|_U = f|_U$. Write

$$D_{f_V\psi_V} := D_V - \sqrt{-1}c(v^{f_V\psi_V}).$$

Write $D_V = D_{G/K}^V + D_{N_V}$ as in (4.6). Then Lemma 4.6 applies directly to the corresponding operators on V .

4.5 Operators on M and V

The reason for the constructions in Subsection 4.4 is that the manifold V is complete, so that the operators

$$D_{f_V\psi_V}^2 + 1 \tag{4.10}$$

and

$$D_{N_V}D_{N_V}^* + f_V^2\|v^{\psi_V}\|^2 + 1$$

are symmetric, and norm-increasing. This implies that they are invertible, with bounded inverses with norms at most 1. (This is generally not true

for the operator $D_{f\psi}|_{\mathcal{U}}^2 + 1$ on $L^2(\mathcal{E}|_{\mathcal{U}})$, for example.) Furthermore, after restriction to \mathcal{U} , the above operators are equal to

$$(D_{f\psi}^2 + 1)|_{\mathcal{U}} \quad (4.11)$$

and

$$(D_N D_N^* + f^2 \|v^\psi\|^2 + 1)|_{\mathcal{U}},$$

respectively. In Lemma 5.5, we will deduce an estimate on \mathcal{U} for the inverse of the operator $D_{f\psi}^2 + 1$ from the corresponding estimate for the inverse of (4.10), using the following relation between these inverses.

Lemma 4.8. *Let $\varphi_1, \varphi_2 \in \text{End}(\mathcal{E})^G$ be supported inside \mathcal{U} . Then there are G -equivariant bounded operators T_0 and T_1 on $L^2(\mathcal{E}|_{\mathcal{U}})$, with norms at most 1, and there is $\varphi \in \text{End}(\mathcal{E})^G$, supported in \mathcal{U} , such that for $\alpha \in \{0, 1\}$,*

$$\begin{aligned} & \varphi_1 D_{f\psi}^\alpha (D_{f\psi}^2 + 1)^{-1} \varphi_2 - \varphi_1 D_{f_V\psi_V}^\alpha (D_{f_V\psi_V}^2 + 1)^{-1} \varphi_2 \\ &= \varphi_1 T_0 \varphi (D_{f_V\psi_V}^2 + 1)^{-1} \varphi_2 + \varphi_1 T_1 \varphi D_{f_V\psi_V} (D_{f_V\psi_V}^2 + 1)^{-1} \varphi_2. \end{aligned} \quad (4.12)$$

Proof. Let $\chi \in C^\infty(\mathcal{U})$ be cocompactly supported, such that $\chi \equiv 1$ on $\text{supp } \varphi_1 \cup \text{supp } \varphi_2$. Then

$$\varphi_1 D_{f\psi}^\alpha (D_{f\psi}^2 + 1)^{-1} \varphi_2 = \varphi_1 D_{f\psi}^\alpha (D_{f\psi}^2 + 1)^{-1} \chi (D_{f_V\psi_V}^2 + 1) (D_{f_V\psi_V}^2 + 1)^{-1} \varphi_2. \quad (4.13)$$

Similarly, because $\varphi_1 D_{f_V\psi_V}^\alpha = \varphi_1 D_{f\psi}^\alpha$,

$$\varphi_1 D_{f_V\psi_V}^\alpha (D_{f_V\psi_V}^2 + 1)^{-1} \varphi_2 = \varphi_1 D_{f\psi}^\alpha (D_{f\psi}^2 + 1)^{-1} (D_{f\psi}^2 + 1) \chi (D_{f_V\psi_V}^2 + 1)^{-1} \varphi_2. \quad (4.14)$$

Here we have used the fact that, while the operators $D_{f\psi}$ and $D_{f_V\psi_V}$ act on different spaces, they both act on sections of $\mathcal{E}|_{\mathcal{U}}$. So all compositions in (4.13) and (4.14) are well-defined. Taking the difference of (4.13) and (4.14), we find that the left hand side of (4.12) equals

$$\varphi_1 D_{f\psi}^\alpha (D_{f\psi}^2 + 1)^{-1} (\chi (D_{f_V\psi_V}^2 + 1) - (D_{f\psi}^2 + 1) \chi) (D_{f_V\psi_V}^2 + 1)^{-1} \varphi_2. \quad (4.15)$$

Now $\chi (D_{f_V\psi_V}^2 + 1) = \chi (D_{f\psi}^2 + 1)$, so

$$\begin{aligned} & \chi (D_{f_V\psi_V}^2 + 1) - (D_{f\psi}^2 + 1) \chi = [\chi, D_{f\psi}^2] \\ &= D_{f\psi} [\chi, D_{f\psi}] + [\chi, D_{f\psi}] D_{f\psi} \\ &= -D_{f\psi} c(d\chi) - c(d\chi) D_{f\psi} \\ &= -D_{f\psi} c(d\chi) - c(d\chi) D_{f_V\psi_V}. \end{aligned}$$

Hence (4.15) equals

$$\begin{aligned} & -\varphi_1 D_{f\psi}^{\alpha+1} (D_{f\psi}^2 + 1)^{-1} c(d\chi) (D_{f_V\psi_V}^2 + 1)^{-1} \varphi_2 \\ & -\varphi_1 D_{f\psi}^\alpha (D_{f\psi}^2 + 1)^{-1} c(d\chi) D_{f_V\psi_V} (D_{f_V\psi_V}^2 + 1)^{-1} \varphi_2. \end{aligned}$$

So the claim follows, with

$$\begin{aligned} T_\beta &:= D_{f\psi}^{\alpha+1-\beta} (D_{f\psi}^2 + 1)^{-1} \quad (\text{for } \beta \in \{0, 1\}); \\ \varphi &:= -c(d\chi). \end{aligned}$$

□

5 Local estimates

After the preparations in Section 4, we are ready to prove Theorem 3.12 and Proposition 3.13.

Let $U \subset M$ be a relatively cocompact, G -invariant open subset of M , on which $\|v^\psi\|$ has a positive lower bound. Suppose that U has the properties in Subsection 4.4, i.e. ∂U is smooth and has a tubular neighbourhood in U . Let V , N_V , p_V , ε_V , D_V , f_V and ψ_V be as in Subsection 4.4. Suppose f_V is positive.

5.1 An estimate on V

For any operator A on a Hilbert space, we write $|A| := \sqrt{A^*A}$.

Lemma 5.1. *If S and T are operators on a Hilbert space, with T positive, then*

$$|S(S^*S + T^2 + 1)^{-1}| \leq \frac{1}{2}|T^{-1}|.$$

Proof. For any two real numbers s and t , with $t > 0$, we have

$$\frac{|s|}{s^2 + t^2 + 1} \leq \frac{1}{2t}.$$

This implies the claim for self-adjoint S . For general S , using a polar decomposition gives the desired estimate. □

The following estimate is central to our proof of Proposition 3.15.

Proposition 5.2. *For all $e \in C_c^\infty(G)$ and $h \in C_c^\infty(G/K)$, there is a constant $B_{e,h}$, independent of U , and for all $\varepsilon > 0$, there is a constant $C_{U,\varepsilon,p}$, independent of e and h , such that if*

$$f_V \geq C_{U,\varepsilon,p} \quad (5.1)$$

and

$$f_V^2 \geq C_{U,\varepsilon,p} \|df_V\|, \quad (5.2)$$

then for all $l = 1, \dots, 5$, and $\alpha \in \{0, 1\}$, the operator

$$D_{f_V \psi_V}^\alpha (D_{f_V \psi_V}^2 + 1)^{-1} A_l^V (D_{N_V} D_{N_V}^* + f_V^2 \|v^{\psi_V}\|^2 + 1)^{-1} \pi_G(e) p_V^* h \quad (5.3)$$

is bounded, with norm at most $\varepsilon B_{e,h}$. Here the operators A_l^V are the analogues on V of the operators defined in (4.7).

Proof. Let $e \in C_c^\infty(G)$ and $h \in C_c^\infty(G/K)$ be given. The operators $D_{G/K}^V$, $(D_{G/K}^V)^*$ and $\nabla_{v^{\psi_V}}^{\varepsilon_V}$ satisfy the conditions on the operator A in Proposition 4.4. Indeed, they are G -equivariant G -differential operators. By construction, for these operators the vector bundle endomorphisms a_j and b as in Lemma 4.2 are constant in the $]0, \infty[$ -direction on $\partial U \times]0, \infty[\subset V$, and they are bounded on the compact sets $\overline{U} \cap N$. Therefore, Proposition 4.4 yields a constant $B_{e,h}$, independent of U , and a constant $C_{U,p}$, independent of e and h , such that the operators

$$D_{G/K}^V \pi_G(e) p_V^* h,$$

$$(D_{G/K}^V)^* \pi_G(e) p_V^* h$$

and

$$\nabla_{v^{\psi_V}}^{\varepsilon_V} \pi_G(e) p_V^* h$$

are bounded, with norm at most $C_{U,p} B_{e,h}$. We choose $B_{e,h}$ so that in addition, it is at least equal to $\|e\|_{L^1(G)} \|h\|_\infty$.

Let $\kappa > 0$ be a lower bound for $\|v^{\psi_V}\|$. The function $\|v^{\psi_V}\|$ is G -invariant, hence it takes a maximum on \overline{U} and therefore also on $\partial U \times]0, \infty[$. Let $\|v^{\psi_V}\|_\infty$ be its maximum. We define the number $\|\nabla^{TM} v^{\psi_V}\|_\infty$ in the analogous way. Let $\varepsilon > 0$ be given, and set

$$C_{U,\varepsilon,p} := \max \left\{ \sqrt{\frac{C_{U,p}}{\varepsilon \kappa}}, \frac{C_{U,p}}{2\varepsilon \kappa}, \frac{C_{U,p} \|v^{\psi_V}\|_\infty}{\varepsilon \kappa^2}, \frac{2C_{U,p}}{\varepsilon \kappa^2}, \frac{2 \dim M}{\varepsilon \kappa}, \frac{2 \dim M \|\nabla^{TM} v^{\psi_V}\|_\infty}{\varepsilon \kappa^2} \right\}.$$

Suppose that f_V satisfies (5.1) and (5.2).

We can then prove the desired estimates for the operators (5.3), using the fact that D_{N_V} commutes with $\pi_G(e)$ and $p_V^* h$. Let $s \in L^2(\mathcal{E}_V)$.

1. First of all, we have

$$\begin{aligned}
& \left\| D_{f_V \psi_V}^\alpha (D_{f_V \psi_V}^2 + 1)^{-1} A_1^V (D_{N_V} D_{N_V}^* + f_V^2 \|v^{\psi_V}\|^2 + 1)^{-1} \pi_G(e) p_V^* h s \right\|_{L^2(\mathcal{E}_V)} \\
& \leq \left\| (D_{G/K}^V)^* \pi_G(e) p_V^* h (D_{N_V} D_{N_V}^* + f_V^2 \|v^{\psi_V}\|^2 + 1)^{-1} s \right\|_{L^2(\mathcal{E}_V)} \\
& \leq C_{U,p} B_{e,h} \left\| (f_V^2 \|v^{\psi_V}\|^2 + 1)^{-1} s \right\|_{L^2(\mathcal{E}_V)} \\
& \leq \varepsilon B_{e,h} \|s\|_{L^2(\mathcal{E}|_V)},
\end{aligned}$$

because

$$(f_V^2 \|v^{\psi_V}\|^2 + 1)^{-1} \leq (C_{U,p}/\varepsilon + 1)^{-1} \leq \varepsilon/C_{U,p}.$$

2. Next, note that

$$\begin{aligned}
& \left\| D_{f_V \psi_V}^\alpha (D_{f_V \psi_V}^2 + 1)^{-1} A_2^V (D_{N_V} D_{N_V}^* + f_V^2 \|v^{\psi_V}\|^2 + 1)^{-1} \pi_G(e) p_V^* h s \right\|_{L^2(\mathcal{E}_V)} \\
& \leq \left\| (D_{G/K}^V)^* \pi_G(e) p_V^* h D_{N_V}^* (D_{N_V} D_{N_V}^* + f_V^2 \|v^{\psi_V}\|^2 + 1)^{-1} s \right\|_{L^2(\mathcal{E}_V)} \\
& \leq C_{U,p} B_{e,h} \left\| D_{N_V}^* (D_{N_V} D_{N_V}^* + f_V^2 \|v^{\psi_V}\|^2 + 1)^{-1} s \right\|_{L^2(\mathcal{E}_V)}.
\end{aligned}$$

Since $C_{U,\varepsilon,p}$ is positive, the function f_V does not vanish. By Lemma 5.1, the last expression above is at most equal to

$$C_{U,p} B_{e,h} \frac{1}{2} \left\| f_V^{-1} \|v^{\psi_V}\|^{-1} s \right\|_{L^2(\mathcal{E}_V)} \leq \varepsilon B_{e,h} \|s\|_{L^2(\mathcal{E}|_V)},$$

because

$$f_V^{-1} \leq \frac{2\varepsilon\kappa}{C_{U,p}}.$$

3. Since $(D_{G/K}^V)^*$ is a G -differential operator, it commutes with the G -invariant function f_V . Hence

$$\begin{aligned}
& \left\| D_{f_V \psi_V}^\alpha (D_{f_V \psi_V}^2 + 1)^{-1} A_3^V (D_{N_V} D_{N_V}^* + f_V^2 \|v^{\psi_V}\|^2 + 1)^{-1} \pi_G(e) p_V^* h s \right\|_{L^2(\mathcal{E}_V)} \\
& \leq \left\| c(v^{\psi_V}) (D_{G/K}^V)^* \pi_G(e) p_V^* h f_V (D_{N_V} D_{N_V}^* + f_V^2 \|v^{\psi_V}\|^2 + 1)^{-1} s \right\|_{L^2(\mathcal{E}_V)} \\
& \leq C_{U,p} B_{e,h} \|v^{\psi_V}\|_\infty \left\| f_V (f_V^2 \|v^{\psi_V}\|^2 + 1)^{-1} s \right\|_{L^2(\mathcal{E}_V)} \\
& \leq \varepsilon B_{e,h} \|s\|_{L^2(\mathcal{E}|_V)},
\end{aligned}$$

because

$$f_V (f_V^2 \|v^{\psi_V}\|^2 + 1)^{-1} \leq f_V^{-1} \|v^{\psi_V}\|^{-2} \leq \frac{\varepsilon}{C_{U,p} \|v^{\psi_V}\|_\infty}.$$

4. For all $k = 1, \dots, \dim M$, we have

$$c(e_k)c(\nabla_{e_k}^{\text{TM}} f_V v^{\psi_V}) = e_k(f)c(e_k)c(v^{\psi_V}) + f_V c(e_k)c(\nabla_{e_k}^{\text{TM}} v^{\psi_V}).$$

So we obtain the pointwise estimate

$$\|c(e_k)c(\nabla_{e_k}^{\text{TM}} f_V v^{\psi_V})\| \leq \|df\| \|v^{\psi_V}\| + f_V \|\nabla^{\text{TM}} v^{\psi_V}\|.$$

Denoting the absolute value of operators by $|\cdot|$ as before, we obtain

$$\begin{aligned} & \left| \sum_{k=1}^{\dim M} c(e_k)c(\nabla_{e_k}^{\text{TM}} f_V v^{\psi_V}) (D_{N_V} D_{N_V}^* + f_V^2 \|v^{\psi_V}\|^2 + 1)^{-1} \right| \\ & \leq \dim M (\|df_V\| \|v^{\psi_V}\| + f_V \|\nabla^{\text{TM}} v^{\psi_V}\|) (f_V^2 \|v^{\psi_V}\|^2 + 1)^{-1}. \end{aligned}$$

Therefore,

$$\begin{aligned} & \|D_{f_V \psi_V}^\alpha (D_{f_V \psi_V}^2 + 1)^{-1} A_4^Y (D_{N_V} D_{N_V}^* + f_V^2 \|v^{\psi_V}\|^2 + 1)^{-1} \pi_G(e) p_V^* h s\|_{L^2(\mathcal{E}_V)} \\ & \leq \dim M \left\| (\|df_V\| \|v^{\psi_V}\| + f_V \|\nabla^{\text{TM}} v^{\psi_V}\|) (f_V^2 \|v^{\psi_V}\|^2 + 1)^{-1} \right\| \|e\|_{L^1(G)} \|h\|_\infty \|s\|_{L^2(\mathcal{E}_V)} \\ & \leq \varepsilon B_{e,h} \|s\|_{L^2(\mathcal{E}_V)}, \end{aligned}$$

because

$$\frac{\|df\| \|v^{\psi_V}\|}{f_V^2 \|v^{\psi_V}\|^2 + 1} \leq \frac{\varepsilon}{2 \dim M},$$

and

$$\frac{f_V \|\nabla^{\text{TM}} v^{\psi_V}\|_\infty}{f_V^2 \|v^{\psi_V}\|^2 + 1} \leq \frac{\varepsilon}{2 \dim M}.$$

5. Finally, since $\nabla_{v^{\psi_V}}^{\mathcal{E}_V}$ is a G-differential operator, it commutes with the G-invariant function f_V . Hence

$$\begin{aligned} & \|D_{f_V \psi_V}^\alpha (D_{f_V \psi_V}^2 + 1)^{-1} A_5^Y (D_{N_V} D_{N_V}^* + f_V^2 \|v^{\psi_V}\|^2 + 1)^{-1} \pi_G(e) p_V^* h s\|_{L^2(\mathcal{E}_V)} \\ & 2 \|\nabla_{\psi_V}^\mathcal{E} \pi_G(e) p_V^* h f_V (D_{N_V} D_{N_V}^* + f_V^2 \|v^{\psi_V}\|^2 + 1)^{-1} s\|_{L^2(\mathcal{E}_V)} \\ & \leq 2C_{u,p} B_{e,h} \|(f_V^2 \|v^{\psi_V}\|^2 + 1)^{-1} s\|_{L^2(\mathcal{E}_V)} \\ & \leq \varepsilon B_{e,h} \|s\|_{L^2(\mathcal{E}_V)}, \end{aligned}$$

because

$$(f_V^2 \|v^{\psi_V}\|^2 + 1)^{-1} \leq \frac{\varepsilon}{2C_{u,p}}.$$

□

5.2 An estimate for D_{N_V}

Proposition 5.2 will allow us to deduce estimates for $(D_{f_V}^2 + 1)^{-1}$ from estimates for $(D_{N_V} D_{N_V}^* + f_V^2 \|v^{\psi_V}\|^2 + 1)^{-1}$.

Lemma 5.3. *For all $e \in C_c^\infty(G)$ and $h \in C_c^\infty(G/K)$, there is a constant $B_{e,h}$, and for all $\varepsilon > 0$, there is a constant $C_{u,\varepsilon,p} > 0$, independent of e and h , such such that if*

$$f_V \geq C_{u,\varepsilon,p}$$

then for $\alpha \in \{0, 1\}$, the operator

$$D_{f_V \psi_V}^\alpha (D_{N_V} D_{N_V}^* + f_V^2 \|v^{\psi_V}\|^2 + 1)^{-1} \pi_G(e) p_V^* h \quad (5.4)$$

is bounded, with norm at most $\varepsilon B_{e,h}$.

Proof. Let $e \in C_c^\infty(G)$ and $h \in C_c^\infty(G/K)$ be given. Let $B_{e,h}$ be as in Proposition 4.4, but also at least equal to $\|e\|_{L^1(G)} \|h\|_\infty$. Let $C_{D_{G/K}^V, p}$ be as in Proposition 4.4, for $A = D_{G/K}^V$. Let $\kappa > 0$ be a lower bound for $\|v^{\psi_V}\|$. Let $\varepsilon > 0$ be given, and set

$$C_{u,\varepsilon,p} := \max \left\{ \frac{1}{\sqrt{\varepsilon \kappa}}, \frac{3}{\varepsilon \kappa}, \sqrt{\frac{3C_{D_{G/K}^V, p}}{\varepsilon \kappa^2}} \right\}.$$

Suppose $f_V \geq C_{u,\varepsilon,p}$.

For $\alpha = 0$, the operator (5.4) has norm at most

$$\left\| (f_V^2 \|v^{\psi_V}\|^2 + 1)^{-1} \right\| \|e\|_{L^1(G)} \|h\|_\infty \leq \varepsilon B_{e,h},$$

because $(f_V^2 \|v^{\psi_V}\|^2 + 1)^{-1} \leq \varepsilon$.

Now suppose $\alpha = 1$. Write

$$D_{f_V \psi_V} = D_{N_V} + D_{G/K}^V - \sqrt{-1} f_V c(v^{\psi_V}).$$

By Lemma 5.1, we have

$$\begin{aligned} & \|D_{N_V} (D_{N_V} D_{N_V}^* + f_V^2 \|v^{\psi_V}\|^2 + 1)^{-1} \pi_G(e) p_V^* h\| \\ & \leq \frac{1}{2} \|f_V^{-1} \|v^{\psi_V}\|^{-1}\| \|e\|_{L^1(G)} \|h\|_\infty \leq \varepsilon B_{e,h}/3, \end{aligned}$$

because

$$f_V^{-1} \|v^{\psi_V}\|^{-1} \leq 2\varepsilon/3.$$

Next, because $\pi_G(e)$ and p_V^*h commute with D_{N_V} , we have

$$\begin{aligned} & \|D_{G/K}^V(D_{N_V}D_{N_V}^* + f_V^2\|v^{\psi_V}\|^2 + 1)^{-1}\pi_G(e)p_V^*h\| \\ & \leq C_{D_{G/K}^V,p}B_{e,h}\|(f_V^2\|v^{\psi_V}\|^2 + 1)^{-1}\| \leq \varepsilon B_{e,h}/3, \end{aligned}$$

because

$$(f_V^2\|v^{\psi_V}\|^2 + 1)^{-1} \leq \frac{\varepsilon}{3C_{D_{G/K}^V,p}}.$$

Finally,

$$\begin{aligned} & \|\sqrt{-1}f_Vc(v^{\psi_V})(D_{N_V}D_{N_V}^* + f_V^2\|v^{\psi_V}\|^2 + 1)^{-1}\pi_G(e)p_V^*h\| \\ & \leq \|f_V\|v^{\psi_V}\|(f_V^2\|v^{\psi_V}\|^2 + 1)^{-1}\| \|e\|_{L^1(G)}\|h\|_\infty \leq \varepsilon B_{e,h}/3, \end{aligned}$$

because $f_V\|v^{\psi_V}\|(f_V^2\|v^{\psi_V}\|^2 + 1)^{-1} \leq \varepsilon/3$. \square

5.3 Proof of Proposition 3.15

By combining Lemma 4.8, Proposition 5.2 and Lemma 5.3, we obtain an estimate for the inverse of $D_{f_V}^2 + 1$ on \mathcal{U} . This begins with an estimate for the inverse of $D_{f_V\psi_V}^2 + 1$.

Lemma 5.4. *For all $e \in C_c^\infty(G)$ and $h \in C_c^\infty(G/K)$, there is a constant $B_{e,h}$, and for all $\varepsilon > 0$, and all $\varphi_1, \varphi_2 \in \text{End}(E)^G$, supported in \mathcal{U} , there is a constant $C_{\varphi_1,\varphi_2,\varepsilon,p} > 0$, independent of e and h , such that if*

$$f_V \geq C_{\varphi_1,\varphi_2,\varepsilon,p} \quad (5.5)$$

and

$$f_V^2 \geq C_{\varphi_1,\varphi_2,\varepsilon,p} \|df_V\|, \quad (5.6)$$

then for $\alpha \in \{0, 1\}$, we have

$$\|\varphi_1 D_{f_V\psi_V}^\alpha (D_{f_V\psi_V}^2 + 1)^{-1} \pi_G(e) p^* h \varphi_2\| \leq \varepsilon B_{e,h}. \quad (5.7)$$

Proof. Let $e \in C_c^\infty(G)$ and $h \in C_c^\infty(G/K)$ be given. Let $B_{e,h}$ be at least as large as the constants $B_{e,h}$ in Proposition 5.2 and Lemma 5.3. Let $\varepsilon > 0$ be given. Because φ_1 and φ_2 are G -equivariant and supported in \mathcal{U} , they are bounded operators on $L^2(\mathcal{E})$. Set

$$\varepsilon' := \frac{\varepsilon}{6(\|\varphi_1\| \|\varphi_2\| + 1)}$$

Let $C_{\varphi_1, \varphi_2, \varepsilon, p}$ be the maximum of the constants $C_{\mathbb{U}, \varepsilon', p}$ in Proposition 5.2 and Lemma 5.3.

Suppose that f_V satisfies (5.5) and (5.6). By Lemma 4.6, the norm on the left hand side of (5.7) is at most equal to

$$\begin{aligned} & \left\| \varphi_1 D_{f_V \psi_V}^\alpha (D_{N_V} D_{N_V}^* + f_V^2 \|v^{\psi_V}\|^2 + 1)^{-1} \pi_G(e) p^* h \varphi_2 \right\| \\ & + \left\| \varphi_1 D_{f_V \psi_V}^\alpha (D_{f_V \psi_V}^2 + 1)^{-1} \Delta (D_{N_V} D_{N_V}^* + f_V^2 \|v^{\psi_V}\|^2 + 1)^{-1} \pi_G(e) p^* h \varphi_2 \right\|. \end{aligned} \quad (5.8)$$

Here Δ was defined in (4.8). Lemma 5.3 implies that the first term in (5.8) is at most equal to

$$\|\varphi_1\| \|\varphi_2\| \varepsilon' B_{e, h} \leq \varepsilon B_{e, h} / 6.$$

Proposition 5.2 implies that the second term in (5.8) is at most equal to

$$5 \|\varphi_1\| \|\varphi_2\| \varepsilon' B_{e, h} \leq 5 \varepsilon B_{e, h} / 6,$$

so the claim follows. \square

Using Lemma 5.4, we obtain the estimate for the inverse of $D_{f\psi}^2 + 1$ that we need.

Lemma 5.5. *For all $e \in C_c^\infty(G)$ and $h \in C_c^\infty(G/K)$, there is a constant $B_{e, h}$, and for all $\varepsilon > 0$, and all $\varphi_1, \varphi_2 \in \text{End}(E)^G$, supported in \mathbb{U} , there is a constant $C_{\varphi_1, \varphi_2, \varepsilon, p} > 0$, independent of e and h , such that if*

$$f|_{\mathbb{U}} \geq C_{\varphi_1, \varphi_2, \varepsilon, p} \quad (5.9)$$

and

$$f^2|_{\mathbb{U}} \geq C_{\varphi_1, \varphi_2, \varepsilon, p} \|df|_{\mathbb{U}}\|, \quad (5.10)$$

then for $\alpha \in \{0, 1\}$, we have

$$\left\| \varphi_1 D_{f\psi}^\alpha (D_{f\psi}^2 + 1)^{-1} \pi_G(e) p^* h \varphi_2 \right\| \leq \varepsilon B_{e, h}. \quad (5.11)$$

Proof. Let $e \in C_c^\infty(G)$ and $h \in C_c^\infty(G/K)$ be given. Let $B_{e, h}$ be as in Lemma 5.4. Let $\varphi_1, \varphi_2 \in \text{End}(E)^G$ be supported in \mathbb{U} , and let $\varepsilon > 0$. Let T_0, T_1 and φ be as in Lemma 4.8. We use tildes on the constants in Lemma 5.4 to distinguish them from the constants in this lemma, and set

$$\begin{aligned} \varepsilon' &:= \varepsilon / (\|\varphi_1\| + 1); \\ C_{\varphi_1, \varphi_2, \varepsilon, p} &:= 2 \max\{\tilde{C}_{\varphi_1, \varphi_2, \varepsilon/3, p}, \tilde{C}_{\varphi, \varphi_2, \varepsilon'/3, p}\}. \end{aligned}$$

By Lemma 4.8, the norm on the left hand side of (5.11) is at most equal to

$$\begin{aligned} & \left\| \varphi_1 D_{f_V \psi_V}^\alpha (D_{f_V \psi_V}^2 + 1)^{-1} \pi_G(e) p^* h \varphi_2 \right\| \\ & + \left\| \varphi_1 T_0 \right\| \cdot \left\| \varphi (D_{f_V \psi_V}^2 + 1)^{-1} \pi_G(e) p^* h \varphi_2 \right\| \\ & + \left\| \varphi_1 T_1 \right\| \cdot \left\| \varphi D_{f_V \psi_V} (D_{f_V \psi_V}^2 + 1)^{-1} \pi_G(e) p^* h \varphi_2 \right\|. \end{aligned} \quad (5.12)$$

Suppose that f satisfies (5.9) and (5.10). Then f_V can be chosen so that it satisfies (5.5) and (5.6), with $C_{\varphi_1, \varphi_2, \varepsilon, p}$ in (5.5) and (5.6) replaced by $C_{\varphi_1, \varphi_2, \varepsilon, p}/2$ as chosen in this proof. Then Lemma 5.4 implies that (5.12) is at most equal to

$$\frac{\varepsilon}{3} \left(1 + 2 \frac{\|\varphi_1\|}{\|\varphi_1\| + 1} \right) B_{e,h} \leq \varepsilon B_{e,h}.$$

□

We are now ready to prove Proposition 3.15, and hence Theorem 3.12.

Proof of Proposition 3.15. Let an open cover $\{U_j\}_{j=1}^\infty$ of M and a partition of unity $\{\chi_j\}_{j=1}^\infty$ as in Subsection 3.5 be given. For each $j \geq 1$, consider the vector bundle endomorphisms χ_j and $c(d\chi_j)$ of \mathcal{E} . For each $j \geq 1$, set

$$C_j := \max\{C_{\chi_j, \chi_j, 2^{-j}/3}, C_{c(d\chi_j), \chi_j, 2^{-j}/3}\},$$

with $C_{\chi_j, \chi_j, 2^{-j}/3}$ and $C_{c(d\chi_j), \chi_j, 2^{-j}/3}$ as in Lemma 5.5. Because every point in m lies in only finitely many of the sets U_j , there is a function $\rho \in C^\infty(M)^G$ such that for all $j \geq 1$,

$$\rho|_{U_j} \geq C_j.$$

Suppose that $f \in C^\infty(M)^G$ is ρ -admissible. Then for all j ,

$$\begin{aligned} f|_{U_j} & \geq C_j; \\ f|_{U_j}^2 & \geq C_j \|df|_{U_j}\|. \end{aligned}$$

Let $e \in C_c^\infty(G)$ and $h \in C_c^\infty(G/K)$ be given, and let $B_{e,h}$ be as in Lemma 5.5.

Note that for all j ,

$$(D_{f\psi}^2 + 1)^{-1} \pi_G(e) p^* h \chi_j^2 = \chi_j (D_{f\psi}^2 + 1)^{-1} \pi_G(e) p^* h \chi_j + [(D_{f\psi}^2 + 1)^{-1}, \chi_j] \pi_G(e) p^* h \chi_j.$$

Now

$$[(D_{f\psi}^2 + 1)^{-1}, \chi_j] = (D_{f\psi}^2 + 1)^{-1} (D_{f\psi} c(d\chi_j) + c(d\chi_j) D_{f\psi}) (D_{f\psi}^2 + 1)^{-1}.$$

Therefore,

$$\begin{aligned} \|(D_{f_\psi}^2 + 1)^{-1} \pi_G(e) p^* h \chi_j^2\| &\leq \|\chi_j (D_{f_\psi}^2 + 1)^{-1} \pi_G(e) p^* h \chi_j\| \\ &\quad + \|(D_{f_\psi}^2 + 1)^{-1} D_{f_\psi} c(d\chi_j) (D_{f_\psi}^2 + 1)^{-1} \pi_G(e) p^* h \chi_j\| \\ &\quad + \|(D_{f_\psi}^2 + 1)^{-1} c(d\chi_j) D_{f_\psi} (D_{f_\psi}^2 + 1)^{-1} \pi_G(e) p^* h \chi_j\|. \end{aligned}$$

By Lemma 5.5, all three terms on the right hand side are at most equal to $2^{-j} B_{e,h}/3$ for all $j \geq 1$. Hence Proposition 3.15 follows. \square

By the arguments in Subsection 3.5, Proposition 3.15 implies Theorem 3.12.

5.4 Independence of choices

Let us prove Proposition 3.13. We start by showing that different admissible functions lead to the same index.

Lemma 5.6. *In the setting of Theorem 3.12, let f_0 and f_1 be two ρ -admissible functions. Then*

$$\text{index}_G D_{f_0\psi} = \text{index}_G D_{f_1\psi}.$$

Proof. Let f_0 and f_1 be two ρ -admissible functions. Then for all constants $a \geq 1$ and $j = 0, 1$, the function af_j is ρ -admissible. Furthermore, by a homotopy argument, we have

$$\text{index}_G (D_{af_j\psi}) = \text{index}_G (D_{f_j\psi})$$

For $t \in [0, 1]$, we consider the function

$$h_t := 4((1-t)f_0 + tf_1) \in C^\infty(M)^G.$$

For all $t \in [0, 1]$, we have

$$\begin{aligned} \frac{h_t^2}{\|dh_t\| + h_t + 1} &= \frac{h_t^2}{4((1-t)(\|df_0\| + f_0 + 1/4) + t(\|df_1\| + f_1 + 1/4))} \\ &\geq \frac{4((1-t)^2 f_0^2 + t^2 f_1^2)}{(1-t)(\|df_0\| + f_0 + 1) + t(\|df_1\| + f_1 + 1)} \\ &\geq \frac{2((1-t)^2 f_0^2 + t^2 f_1^2)}{\max_{j=0,1} (\|df_j\| + f_j + 1)} \\ &\geq 2((1-t)^2 \rho + t^2 \rho) \\ &\geq \rho, \end{aligned}$$

so h_t is ρ -admissible for all t .

We conclude that, by operator homotopies,

$$\begin{aligned} \text{index}_G D_{f_0\psi} &= \text{index}_G(D_{h_0\psi}) \\ &= \text{index}_G(D_{h_1\psi}) \\ &= \text{index}_G D_{f_1\psi}. \end{aligned}$$

□

The space of L^2 -sections of \mathcal{E} depends on the Riemannian metric, through the Riemannian density. Therefore, operator homotopies, for operators on a fixed Hilbert space, are not enough to prove the remaining part of Proposition 3.13. We can use an argument modelled on Section 11.2 of [21], however. We use the notation of Proposition 3.13.

Let B_{S^1} be the standard Riemannian metric on TS^1 . Let \mathbb{C}_1 be the complex Clifford algebra with one generator e_1 . Then, as a complex vector space, $\mathbb{C}_1 = \text{span}_{\mathbb{C}}\{1, e_1\} \cong \mathbb{C}^2$. Consider the spinor bundle $\mathcal{E}_{S^1} = S^1 \times \mathbb{C}_1 \rightarrow S^1$. We have the Dirac operator $D_{S^1} = c(e_1) \frac{d}{d\alpha}$, where α is the angle coordinate on S^1 . If $I \subset]0, 2\pi[$ is an open sub-interval, we embed it into S^1 via the map $x \mapsto e^{ix}$. We write B_I , \mathcal{E}_I and D_I for the restrictions to I of B_{S^1} , \mathcal{E}_{S^1} and D_{S^1} , respectively. We set $I_0 :=]0, \pi/2[$ and $I_1 :=]\pi, 3\pi/2[$.

Lemma 5.7. *There is a Riemannian metric \tilde{B} on $S^1 \times M$, a G -equivariant Clifford module $\tilde{\mathcal{E}} \rightarrow S^1 \times M$, a G -invariant Clifford connection $\nabla^{\tilde{\mathcal{E}}}$ on $\tilde{\mathcal{E}}$, and a smooth, G -equivariant map $\tilde{p}: S^1 \times M \rightarrow G/K$, such that for $j = 0, 1$, the metric $\tilde{B}|_{I_j \times M}$ is the product metric of B_{I_j} and B_j , we have*

$$\tilde{\mathcal{E}}|_{I_j \times M} \cong \mathcal{E}_{I_j} \boxtimes \mathcal{E}_j \rightarrow I_j \times M$$

as G -equivariant Clifford modules, the Dirac operator \tilde{D} associated to $\nabla^{\tilde{\mathcal{E}}}$ satisfies

$$\tilde{D}|_{I_j \times M} = D_{I_j} \otimes 1 + 1 \otimes D^{\mathcal{E}_j},$$

and for all $t \in I_j$ and $m \in M$, we have

$$\tilde{p}(t, m) = p_j(m).$$

Proof. As a G -equivariant vector bundle, we take $\tilde{\mathcal{E}} = \mathcal{E}_{S^1} \boxtimes \mathcal{E}$. The metric \tilde{B} , Clifford action on $\tilde{\mathcal{E}}$ and Clifford connection on $\tilde{\mathcal{E}}$ can be constructed using a partition of unity. The map \tilde{p} exists because G/K is G -equivariantly contractible. □

Fix \tilde{B} , $\tilde{\mathcal{E}}$, $\nabla^{\tilde{\mathcal{E}}}$, \tilde{D} and \tilde{p} as in Lemma 5.7. Let $\tilde{\psi}: S^1 \times M \rightarrow \mathfrak{g}$ and $\tilde{f} \in C^\infty(S^1 \times M)^G$ be the pullbacks of ψ and f , respectively. Consider the deformed Dirac operator

$$\tilde{D}_{\tilde{f}\tilde{\psi}} = \tilde{D} - \sqrt{-1}\tilde{f}\tilde{c}(v^{\tilde{\psi}}),$$

with \tilde{c} the Clifford action by $T(S^1 \times M)$ on $\tilde{\mathcal{E}}$.

Lemma 5.8. *There is a positive function $\tilde{p} \in C^\infty(S^1 \times M)^G$ such that if \tilde{f} is \tilde{p} -admissible, the triple*

$$\left(L^2(\tilde{\mathcal{E}}), \frac{\tilde{D}_{\tilde{f}\tilde{\psi}}}{\sqrt{\tilde{D}_{\tilde{f}\tilde{\psi}}^2 + 1}}, \pi_{S^1} \otimes \pi_{G/G/K} \right) \quad (5.13)$$

is a Kasparov $(C(S^1) \otimes C_0(G/K) \rtimes G, \mathbb{C})$ -cycle. Here $\pi_{S^1}: C(S^1) \rightarrow \mathcal{B}(L^2(\tilde{\mathcal{E}}))$ is defined by pointwise multiplication after pulling back to $S^1 \times M$.

Proof. Since S^1 is compact, the set of zeroes of the vector field $v^{\tilde{\psi}}$ is cocompact. Hence Theorem 3.12 implies that there is a function \tilde{p} such that if \tilde{f} is \tilde{p} -admissible, the triple (5.13) is a Kasparov $(C_0(G/K) \rtimes G, \mathbb{C})$ -cycle. Since π_{S^1} commutes with $\pi_{G/G/K}$, the claim follows. \square

Remark 5.9. Lemma 5.8 still holds if X is replaced by any compact manifold X . The authors also expect it to be true if X is noncompact but complete. Then the representation π_X (of $C_0(X)$ in that case) plays a more important role.

Let \tilde{p} be as in Lemma 5.8, and suppose \tilde{f} is \tilde{p} -admissible. Let

$$[\tilde{D}_{\tilde{f}\tilde{\psi}}] \in KK_1(C(S^1) \otimes C_0(G/K) \rtimes G, \mathbb{C})$$

be the class defined by (5.13). Note that this is a class in odd KK-theory. For an open interval $I \subset]0, 2\pi[$, consider the restriction map

$$r_I: KK_1(C(S^1) \otimes C_0(G/K) \rtimes G, \mathbb{C}) \rightarrow KK_1(C_0(I) \otimes C_0(G/K) \rtimes G, \mathbb{C}),$$

induced by the inclusion map $C_0(I) \hookrightarrow C(S^1)$. We also have the suspension isomorphism

$$s_I: KK_1(C_0(I) \otimes C_0(G/K) \rtimes G, \mathbb{C}) \xrightarrow{\cong} KK_0(C_0(G/K) \rtimes G, \mathbb{C}),$$

see Definition 9.5.6 in [21]. The core of the proof of Proposition 3.13 is the fact that we can recover G -indices of deformed Dirac operators from operators on $S^1 \times M$, using the suspension isomorphism.

Lemma 5.10. *For all open intervals $I \subset]0, 2\pi[$, and $j = 0, 1$, we have*

$$s_I[D_I \otimes 1 + 1 \otimes D_{f_j\psi}^{\varepsilon_j}] = \text{index}_G^{p_j}(D_{f_j\psi}^{\varepsilon_j}) \in \text{KK}_0(C_0(G/K) \rtimes G, \mathbb{C}).$$

Proof. As in Lemmas 9.5.7 and 9.5.8, Exercise 10.9.7 and Proposition 11.2.5 in [21], we have

$$\begin{aligned} s_I[D_I \otimes 1 + 1 \otimes D_{f_j\psi}^{\varepsilon_j}] &= s_I[D_I] \otimes \text{index}_G^{p_j}(D_{f_j\psi}^{\varepsilon_j}) \\ &= \text{index}_G^{p_j}(D_{f_j\psi}^{\varepsilon_j}). \end{aligned}$$

□

Proof of Proposition 3.13. Since S^1 is compact, we can choose f so that \tilde{f} is $\tilde{\rho}$ -admissible. Furthermore, we can choose f so that in addition, it is $\max(\rho_0, \rho_1)$ -admissible. Then by Lemma 5.6, we have for $j = 0, 1$,

$$\text{index}_G^{p_j}(D_{f_j\psi}^{\varepsilon_j}) = \text{index}_G^{p_j}(D_{f\psi}^{\varepsilon_j}). \quad (5.14)$$

Set $I :=]0, 2\pi[$. Consider the class

$$r_I[\tilde{D}_{\tilde{f}\tilde{\psi}}] \in \text{KK}_1(C_0(I) \otimes C_0(G/K) \rtimes G, \mathbb{C}).$$

The inclusion maps $I_j \hookrightarrow I$ are homotopy equivalences. Therefore, after identifying $I_j \cong I$, we have

$$\begin{aligned} [D_I \otimes 1 + 1 \otimes D_{f\psi}^{\varepsilon_0}] &= r_{I_0}[\tilde{D}_{\tilde{f}\tilde{\psi}}] \\ &= r_I[\tilde{D}_{\tilde{f}\tilde{\psi}}] \\ &= r_{I_1}[\tilde{D}_{\tilde{f}\tilde{\psi}}] \\ &= [D_I \otimes 1 + 1 \otimes D_{f\psi}^{\varepsilon_1}] \\ &\in \text{KK}_1(C_0(I) \otimes C_0(G/K) \rtimes G, \mathbb{C}). \end{aligned}$$

Hence, by Lemma 5.10,

$$\text{index}_G^{p_0}(D_{f\psi}^{\varepsilon_0}) = \text{index}_G^{p_1}(D_{f\psi}^{\varepsilon_1}).$$

Combined with (5.14), this implies that

$$\text{index}_G^{p_0}(D_{f_0\psi}^{\varepsilon_0}) = \text{index}_G^{p_1}(D_{f_1\psi}^{\varepsilon_1}).$$

□

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